Diameter of polytopes and The Hirsch Conjecture

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MDS Summer Schhol, Döllnsee — August 14–16, 2012

Hirsch Wars Trilogy

- Episode I: The Phantom Conjecture.
- Episode II: Attack of the Prismatoids.
- Episode III: Revenge of the Linear Bound.

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Hirsch Wars Episode I
The Phantom Conjecture

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Polyhedra and polytopes

The dimension of P is the dimension of its affine hull.

Polyhedra and polytopes

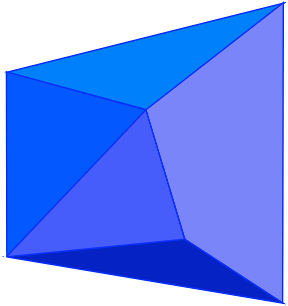
Definition

A (convex) polyhedron P is the intersection of a finite family of

Polyhedra and polytopes

Definition

A (convex) polytope P is the convex hull of a finite set of points in $\mathbb{R}^d$.

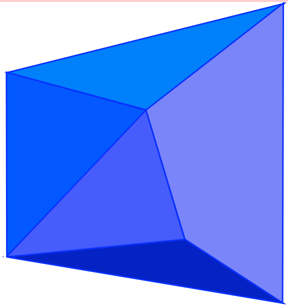


The dimension of P is the dimension of its affine hull.

Polyhedra and polytopes

Polytope = bounded polyhedron.

Every polytope is a polyhedron, every bounded polyhedron is a polytope.



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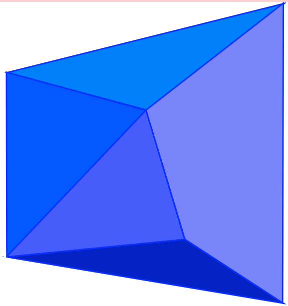

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Faces of *P*

Let *P* be a polytope (or polyhedron) and let *H* be a hyperplane
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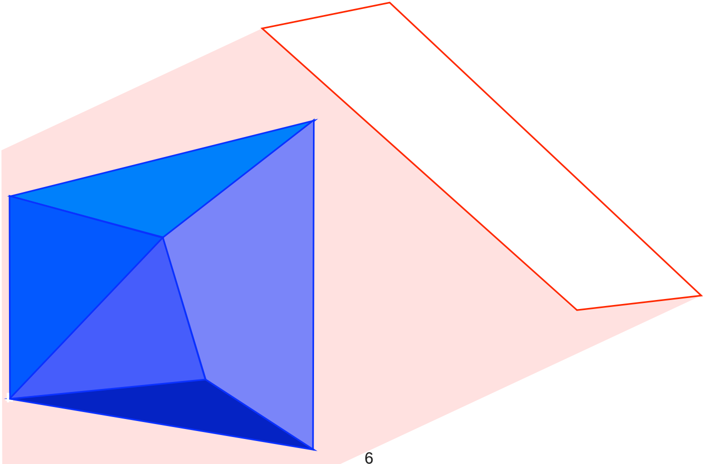

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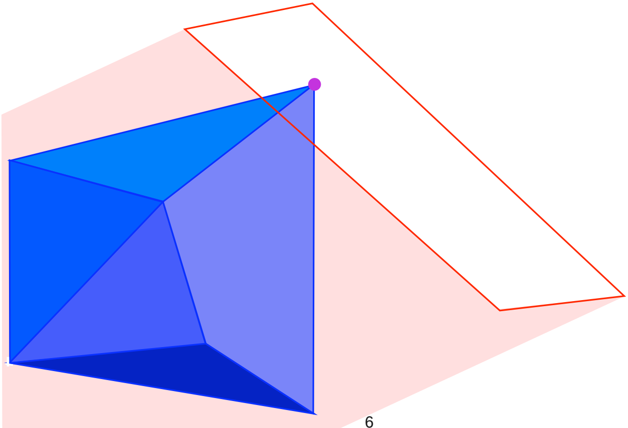
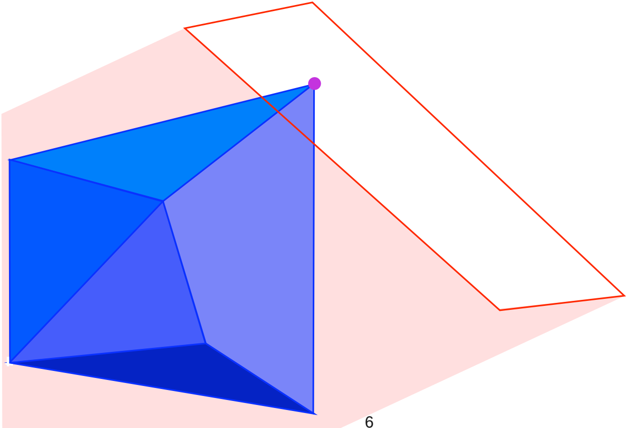


Faces of P

We say that $H \cap P$ is a face of P.



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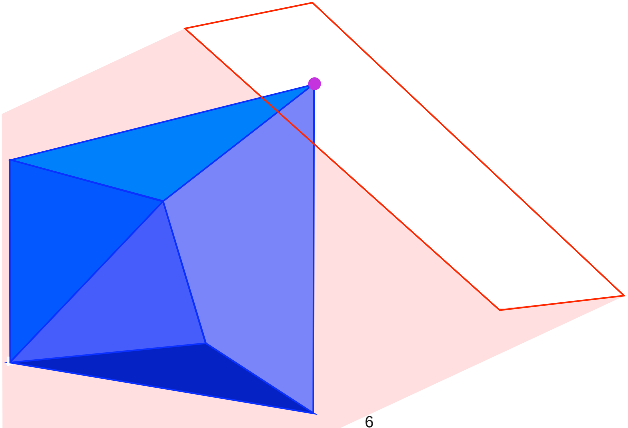
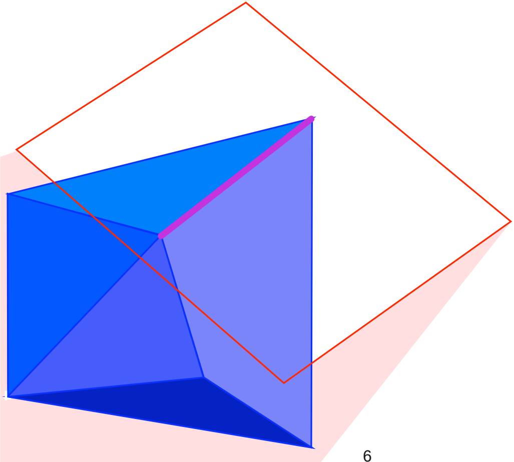
Faces of dimension 0 are called vertices.



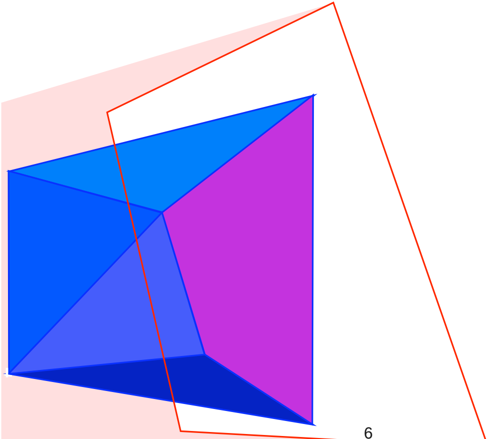
Faces of P

Faces of dimension 1 are called edges.



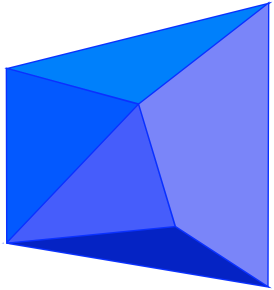
Faces of P

Faces of dimension $d - 1$ are called facets.



The graph of a polytope

Vertices and edges of a polytope P form a graph (finite, undirected)



The distance $d(u, v)$ between vertices u and v is the length (number of edges) of the shortest path from u to v.

For example, $d(u, v) = 2$.

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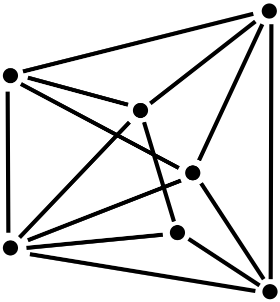

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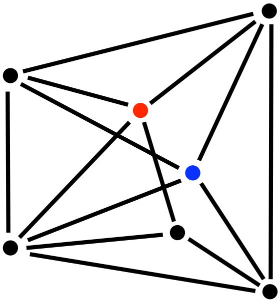

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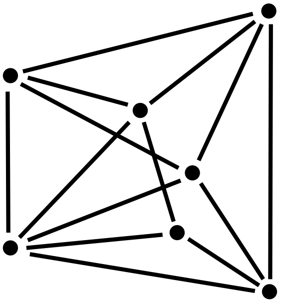

The diameter of $G(P)$ (or of P) is the maximum distance among its vertices:

$$\text{diam}(P) = \max\{d(u, v) : u, v \in V\}.$$

The Hirsch conjecture

Conjecture: Warren M. Hirsch (1957)

For every polytope P with n facets and dimension d,

$$\operatorname{diam}(P) \leq n - d.$$

polytope	facets	dimension	$n-d$	diameter
cube	6	3	3	3
dodecahedron	12	3	9	5
octahedron	8	3	5	2
k-prism	$k+2$	3	$k-1$	$\lfloor k/2 \rfloor + 1$
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Brief history of the conjecture

1. It was communicated by W. M. Hirsch to G. Dantzig in 1957 (Dantzig had recently invented the simplex method for linear programming).
2. Several special cases have been proved: $d \leq 3$, $n - d \leq 6$, $0/1$-polytopes, ...
3. But in the general case **we do not even know of a polynomial bound** for diam(P) in terms of n and d.
4. In 1967, Klee and Walkup disproved the unbounded case.
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“As simple as possible”

Definition

A d-polytope/polyhedron is simple if at every vertex exactly d facets meet. ($\simeq$ facet-defining hyperplanes are “in general position”).

A d-polytope is simplicial if every facet has exactly d vertices. That is, if every proper face is a simplex. ($\simeq$ vertices are “in general position”).

Of course, the (polar) dual of a simple polytope is simplicial, and vice-versa.

Lemma (Klee 1964)

For every n and d the maximum diameter of d-polytopes / d-polyhedra with n facets is achieved at a simple one.

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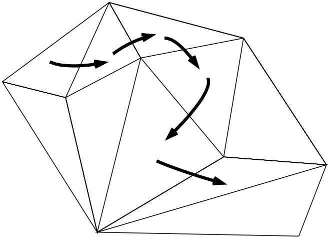
We will often dualize the diameter problem. We want to travel from one facet to another of a polytope Q (the polar of P) along the "dual graph" whose edges correspond to *ridges* of Q.



By the previous lemma we can restrict our attention to simplicial polytopes, whose face lattice is a *simplicial complex* with the topology of a $(d-1)$-sphere. (A *simplicial* $(d-1)$-*sphere*).

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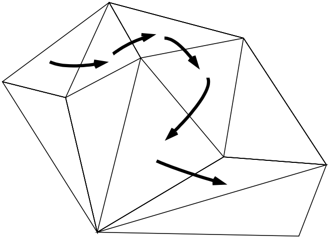

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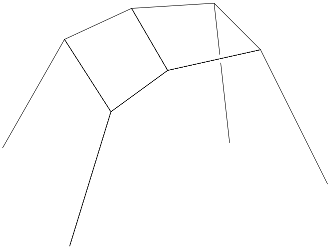

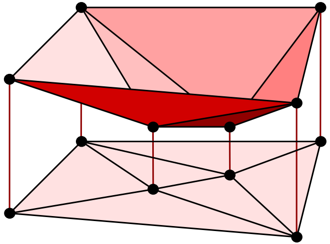

The polar of an unbounded d-polyhedron with n facets "is" a regular triangulation of n points in $\mathbb{R}^{d-1}$.

Linear programming

A linear program is the problem of maximization (or minimization) of a linear functional subject to linear inequality constraints. That is:

Given

- a system $Mx \leq b$ of linear inequalities ($b \in \mathbb{R}^n$, $M \in \mathbb{R}^{d \times n}$), and
- an objective function $c^t \in \mathbb{R}^{d*}$

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- $\max\{c^t \cdot x : x \in \mathbb{R}^d, Mx \leq b\}$ (and a point x where the maximum is attained).

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Linear programming

> *If one would take statistics about which mathematical problem is using up most of the computer time in the world, then (not including database handling problems like sorting and searching) the answer would probably be linear programming.*
>
> (László Lovász, 1980)

A brief history of linear programming

- It was invented in the 1940's by G. Dantzig, L. Kantorovich and J. von Neumann.
- In particular, in 1947 G. Dantzig devised the simplex method: The first practical algorithm for solving linear programs (and still the one most used).
- Around 1980 two polynomial time algorithms for linear programming were proposed by Khachiyan and Karmakar (*ellipsoid* and *interior point* method).
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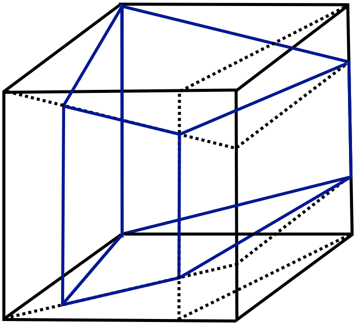

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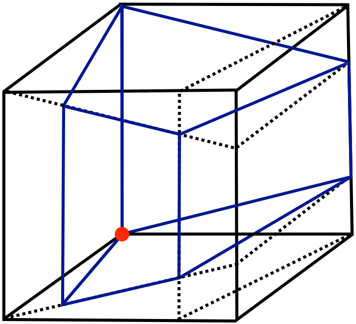

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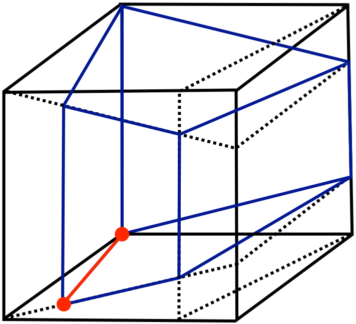

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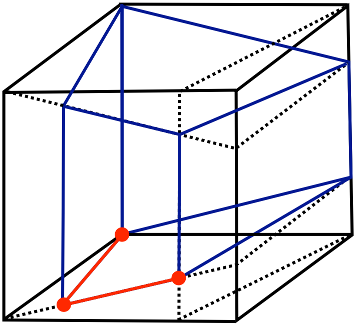

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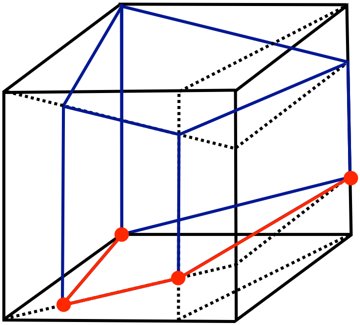

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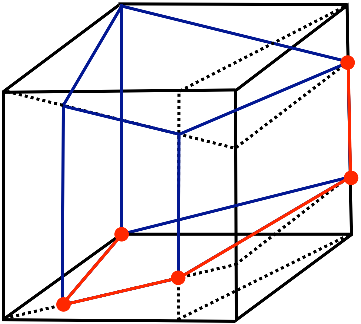

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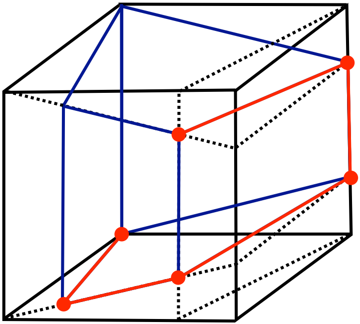

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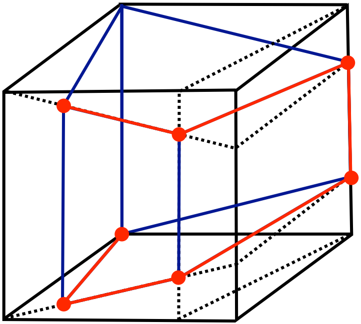

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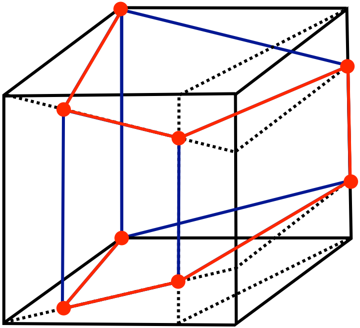

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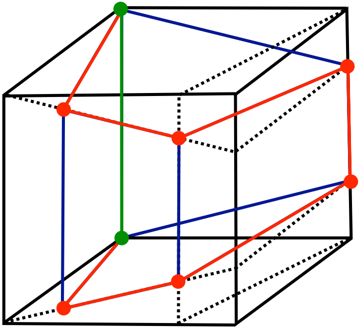

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Connection to the Hirsch conjecture

- The set of feasible solutions $P = \{x \in \mathbb{R}^d : Mx \leq b\}$ is a polyhedron P with (at most) n facets and d dimensions.
- The optimal solution (if it exists) is always attained at a vertex.
- The simplex method [Dantzig 1947] solves the linear program by starting at any feasible vertex and moving along the graph of P, in a monotone fashion, until the optimum is attained.
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In this sense, more important than the standard Hirsch conjecture (which is false) is the following "polynomial version" of it:

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- It holds with equality in simplices ($n = d + 1$, $\delta = 1$) and cubes ($n = 2d$, $\delta = d$).
- If P and Q satisfy it, then so does $P \times Q$: $\delta(P \times Q) = \delta(P) + \delta(Q)$.
- For every $n \leq 2d$, there are polytopes in which the bound is tight (products of simplices).
  We call these "Hirsch-sharp" polytopes.
- For every $n > d$, it is easy to construct unbounded polyhedra where the bound is tight.
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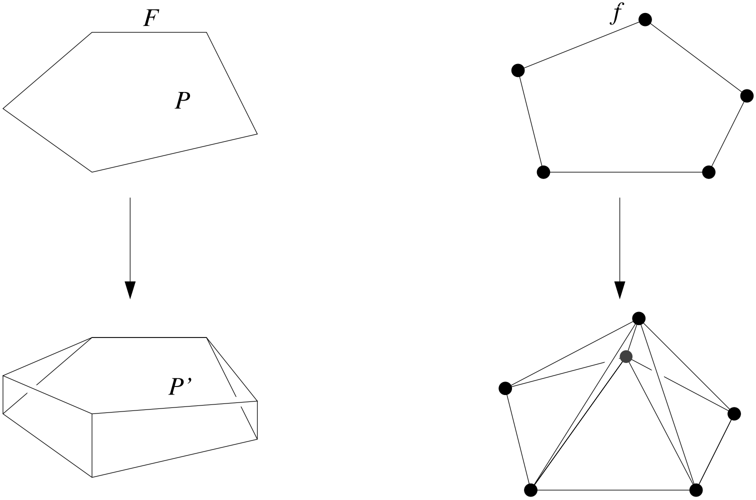

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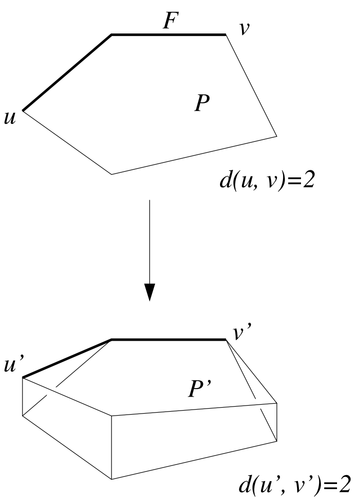

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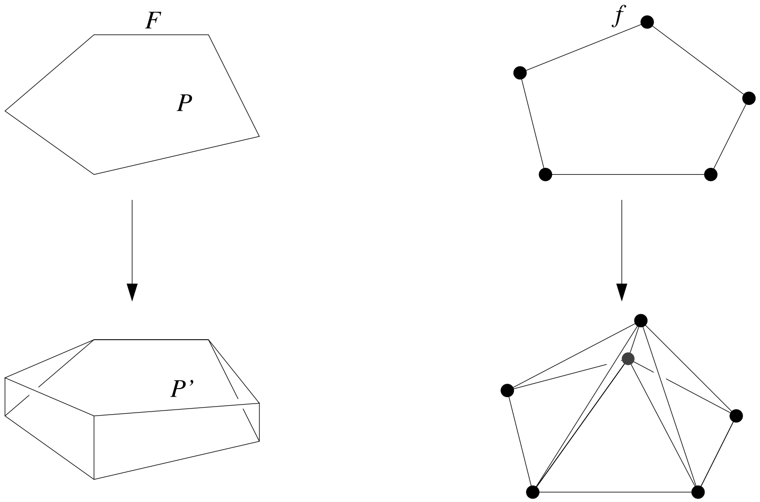

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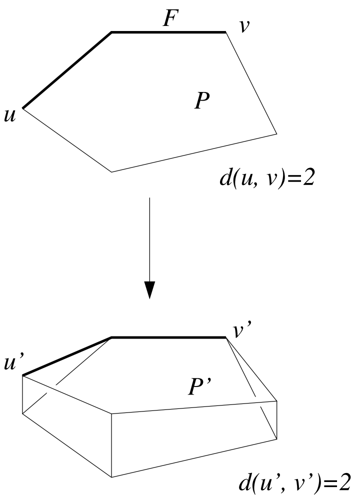

Two important remarks

The d-step Theorem follows from and implies (respectively) the following:

Lemma

For every d-polytope P with n facets and diameter δ there is a $d+1$-polytope with one more facet and the same diameter δ.

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The feasible region of a linear program can be an unbounded polyhedron, instead of a polytope.

Unbounded version of the Hirsch conjecture:

The diameter of any polyhedron P with dimension d and n facets is at most $n - d$.

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Any of these three versions (combinatorial, monotone, unbounded) would imply the Hirsch conjecture...

... but the three were known to be false (although all known counter-examples are only by a linear factor):

- There are unbounded polyhedra of dimension 4 with 8 facets and diameter 5 [Klee-Walkup, 1967].
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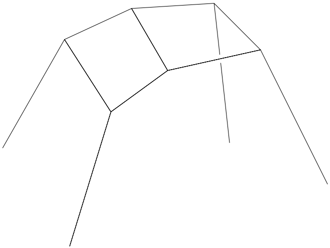

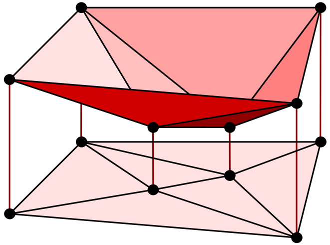

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Remember that

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So, it suffices to show that:

Theorem

There is a regular triangulation of a 4-polytope with 8 vertices that has two tetrahedra at distance five.

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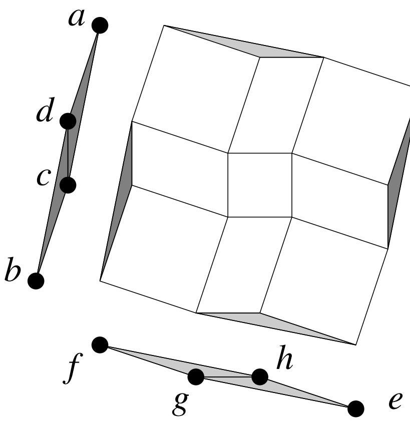
This is a (Cayley Trick view of a) 3D triangulation with 8 vertices and diameter 5:



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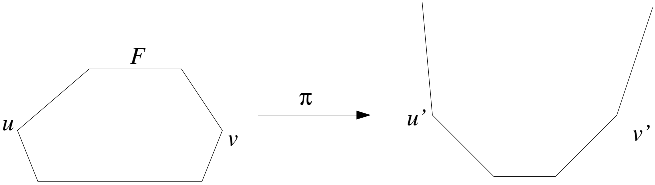

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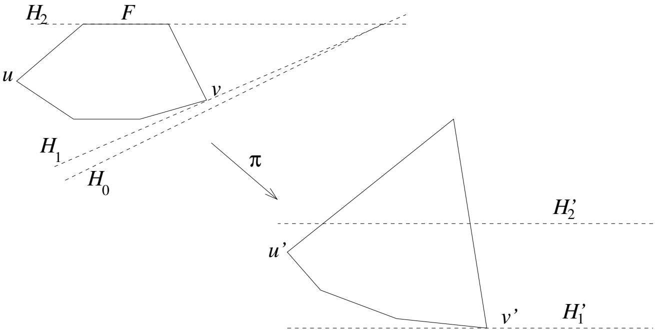

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Mani and Walkup constructed a simplicial 3-ball with 16 vertices and with two tetrahedra *abcd* and *mnop* with the property that any path from *abcd* to *mnop* must revisit a vertex previously abandonded.

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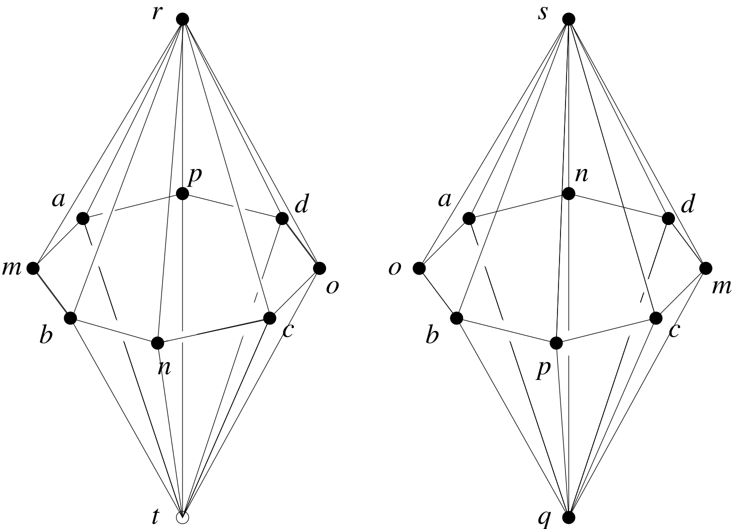

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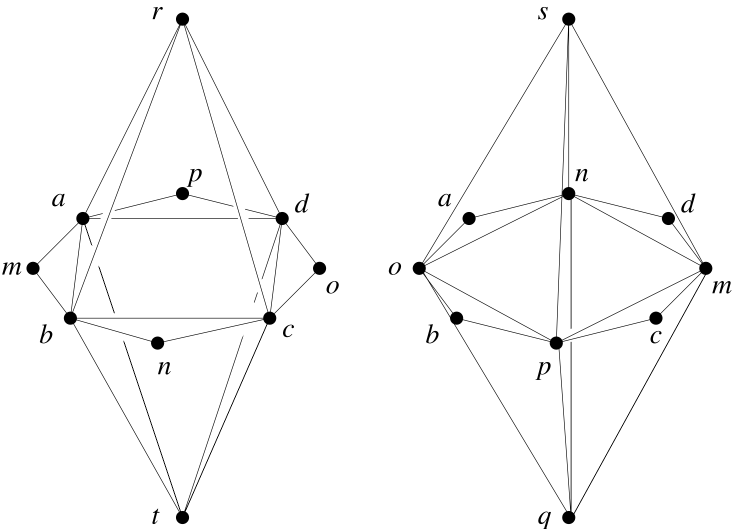

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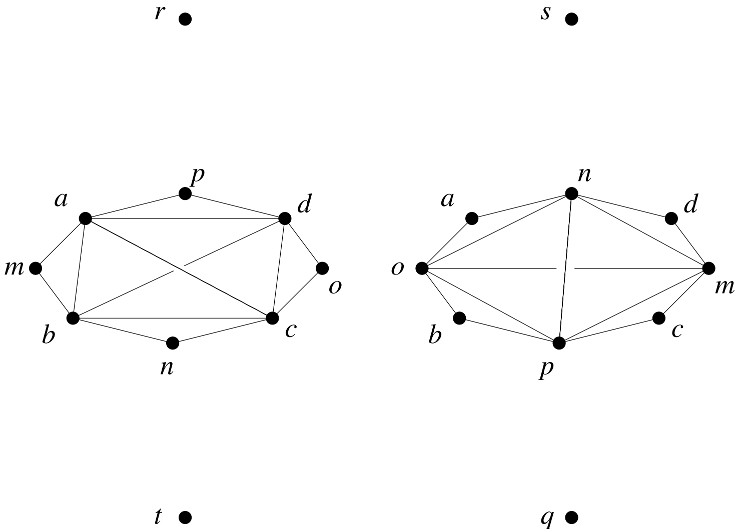

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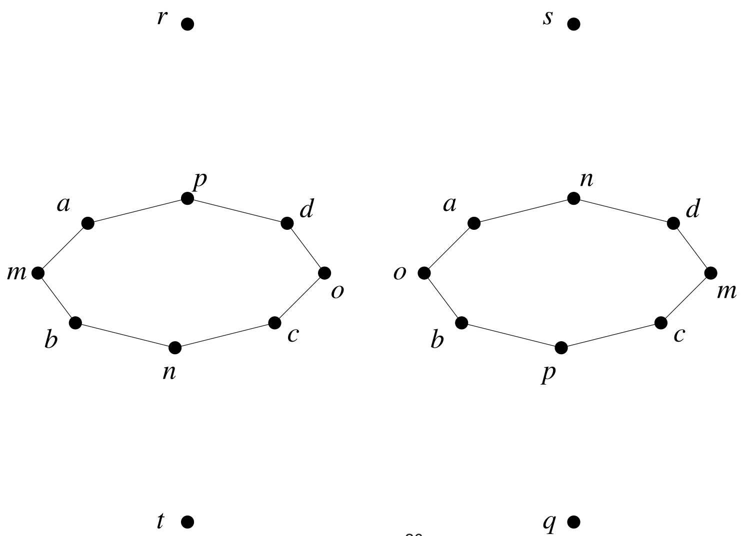

Thank you

TO BE CONTINUED