Hirsch Wars Episode II
Attack of the Prismatoids

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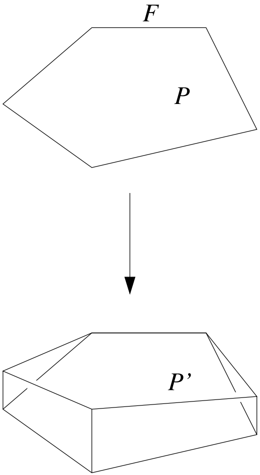

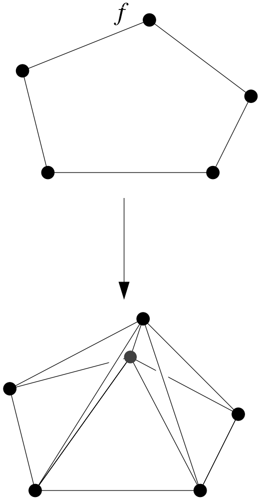

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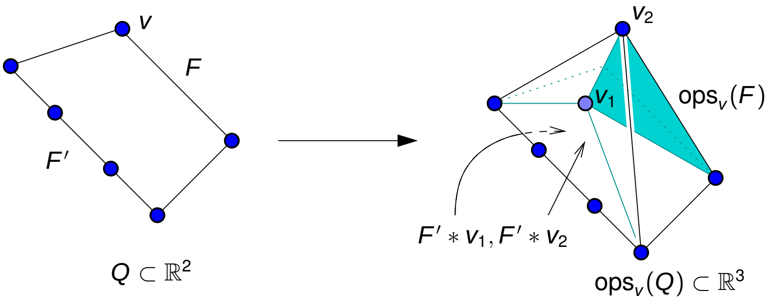

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The strong d-step Theorem is the following modification of it:

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For every d-spindle P with n facets and length λ there is a $d+1$-spindle with one more facet and length $\lambda+1$.

Attack of the prismatoids

The construction of counter-examples to the Hirsch conjecture has two ingredients:

1. A strong *d*-step theorem for spindles/prismatoids.
2. The construction of a prismatoid of dimension 5 and "width" 6.

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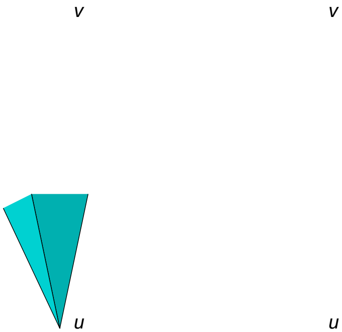
Definition

A *spindle* is a polytope P with two distinguished vertices u and v such that every facet contains either u or v (but not both).



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The *length* of a spindle is the graph distance from u to v.

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3-spindles have length ≤ 3.

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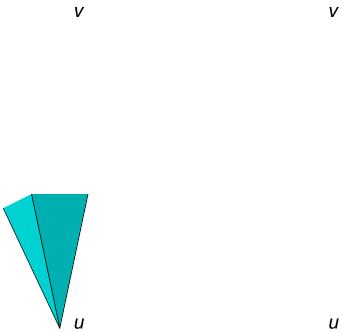

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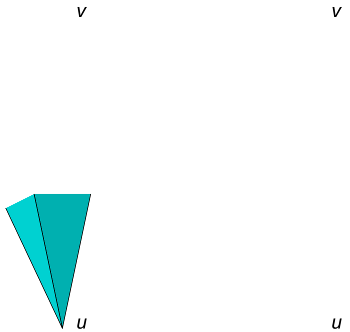

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Theorem (Strong d-step theorem for spindles)

Let P be a spindle of dimension d, with $n > 2d$ facets and length λ. Then there is another spindle P' of dimension $d + 1$, with $n + 1$ facets and length $\lambda + 1$.

That is: we can increase the dimension, length and number of facets of a spindle, all by one, until $n = 2d$.

Corollary

In particular, if a spindle P has length $> d$ then there is another spindle P' (of dimension $n - d$, with $2n - 2d$ facets, and length $\geq \lambda + n - 2d > n - d$) that violates the Hirsch conjecture.

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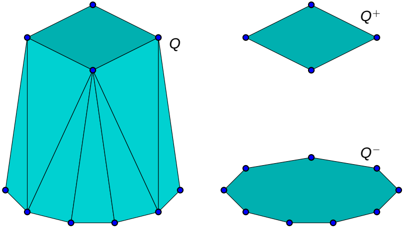

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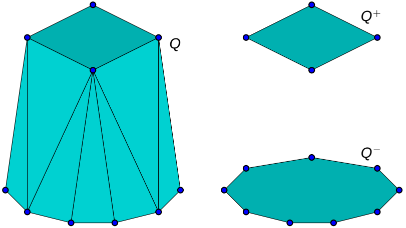

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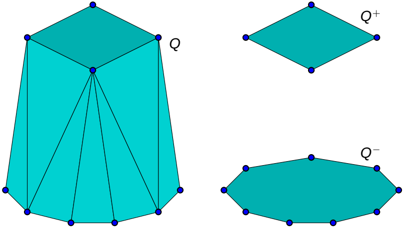

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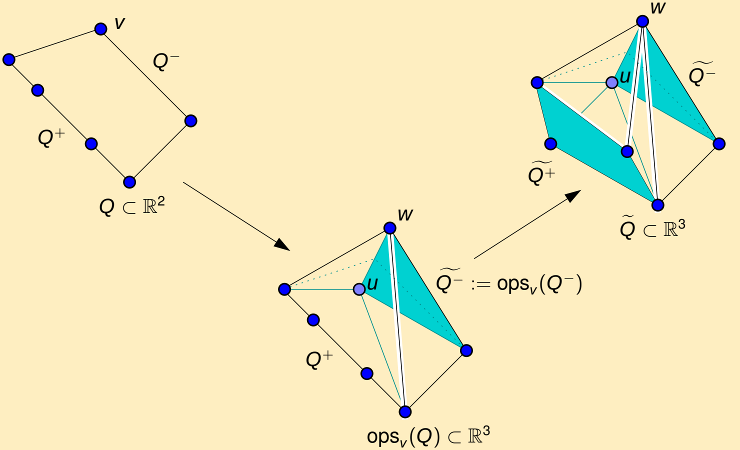
d-step theorem for prismatoids

Proof.



□

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So, to disprove the Hirsch Conjecture we only need to find a prismatoid of dimension d and width larger than d. *Its number of vertices and facets is irrelevant...*

Question

Do they exist?

- 3-prismatoids have width at most 3 (exercise).
- 4-prismatoids have width at most 4 [S.-Stephen-Thomas, 2011].
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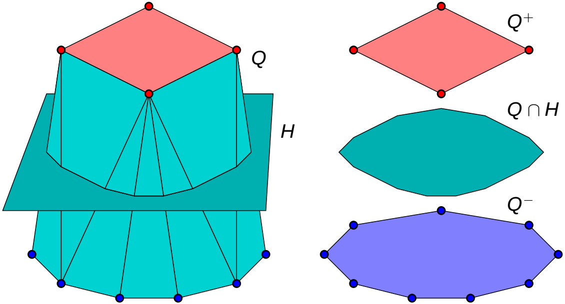
Combinatorics of prismatoids

Analyzing the combinatorics of a d-prismatoid Q can be done via an intermediate slice ...



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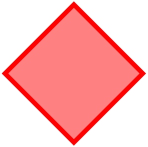
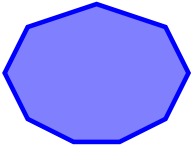
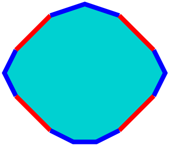
$\frac{1}{2}$  $+ \frac{1}{2}$  $=$ 

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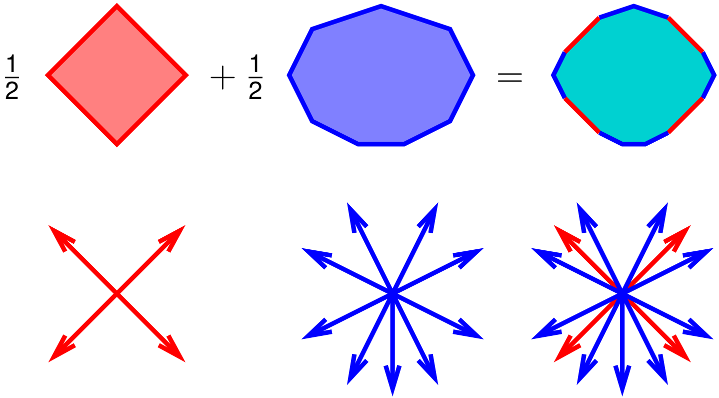
...which equals the Minkowski sum $Q^+ + Q^-$ of the two bases Q^+ and Q^-. The normal fan of $Q^+ + Q^-$ equals the "superposition" of those of Q^+ and Q^-.



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So: the combinatorics of Q follows from the superposition of the normal fans of Q^+ and Q^-.

Remark

The normal fan of a $d-1$-polytope can be thought of as a (geodesic, polytopal) cell decomposition ("map") of the $d-2$-sphere.

Theorem

Let Q be a d-prismatoid with bases Q^+ and Q^- and let G^+ and G^- be the corresponding maps in the $(d-2)$-sphere (central projection of the normal fans of Q^+ and Q^-). Then, the width of Q equals 2 plus the minimum number of steps needed to go from a vertex of G^+ to a vertex of G^- in the (graph of) the superposition of the two maps.

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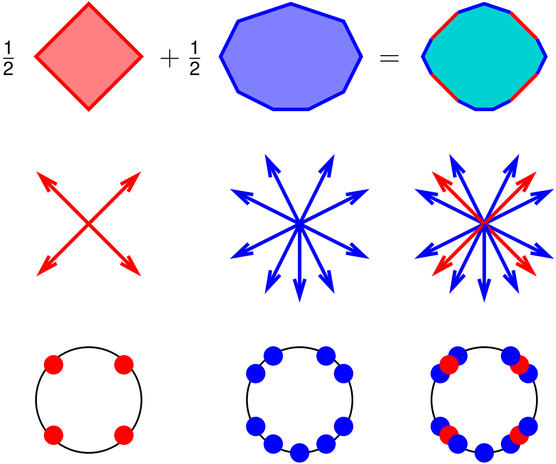

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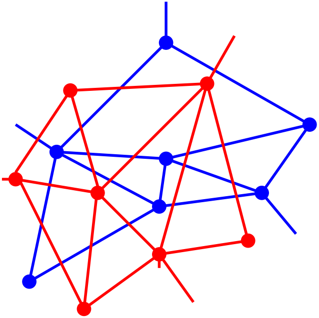

4-prismatoid of width > 4

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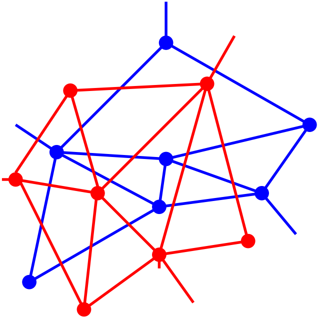

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Example: The Klee-Walkup (unbounded) 4-spindle

Remember that Klee and Walkup, in 1967, disproved the Hirsch conjecture:

Theorem 2 (Klee-Walkup 1967)

There is an unbounded 4-polyhedron with 8 facets and diameter 5.

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The Klee-Walkup polytope is an "unbounded 4-spindle".

What is the corresponding "transversal pair of (geodesic, polytopal) maps"?

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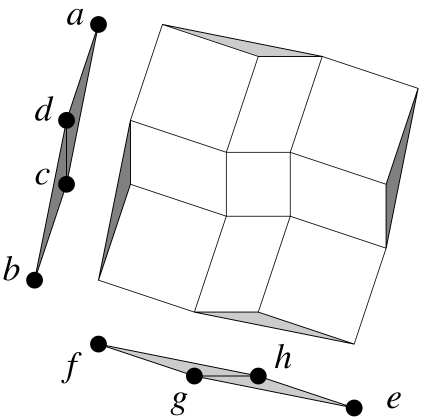

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"Non-Hirsch" 4-prismatoids do not exist:

Theorem (S.-Stephen-Thomas, 2011)

In every transversal pair of maps in the sphere there is a path of length two from some blue vertex to some red vertex.

That is to say:

Corollary (S.-Stephen-Thomas, 2011)

Every prismatoid of dimension 4 has width (at most) four.

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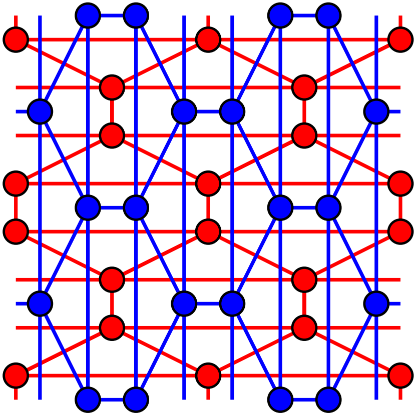

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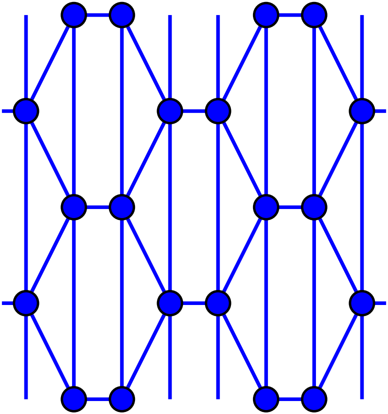

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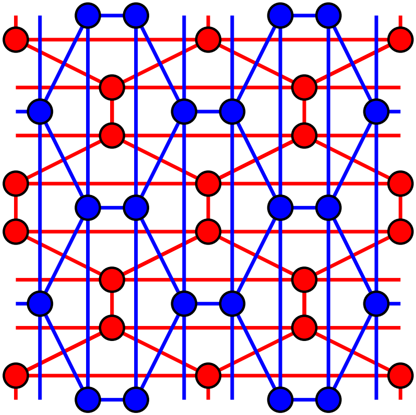

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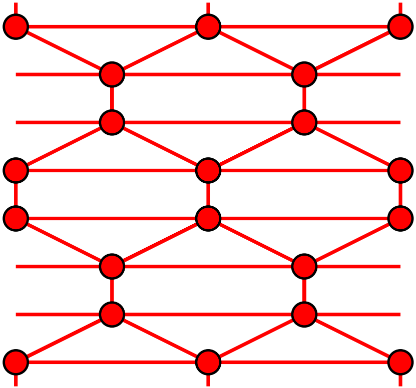

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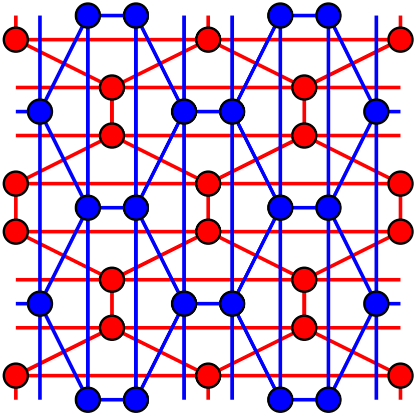

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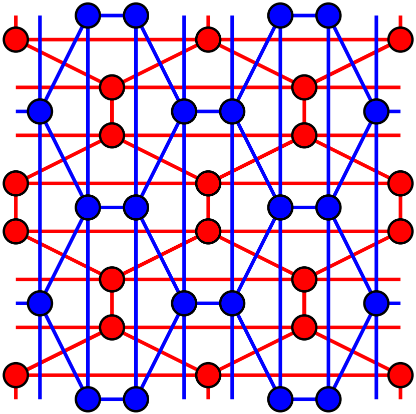

...or with finite ones in the torus!

5-prismatoids of width > 5

To construct 5-dimensional prismatoids we should look at "pairs of maps" in the 3-sphere.

That is, we want a pair of (geodesic, polytopal) cell decompositions of the 3-sphere such that if we draw them one on top of the other (common refinement) there is no path of length ≤ 3 from a blue vertex to a red vertex.

Main idea: If non-Hirsch pairs of maps exist in the torus we should have room enough to construct it in the sphere too . . .

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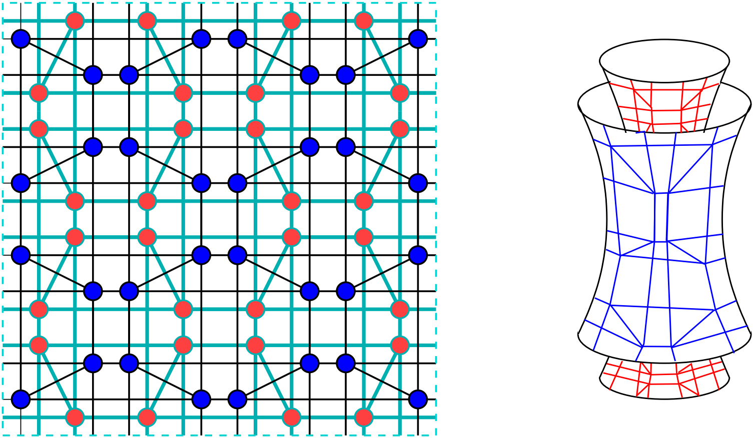

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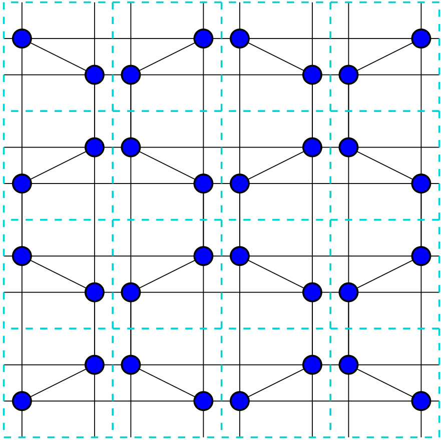

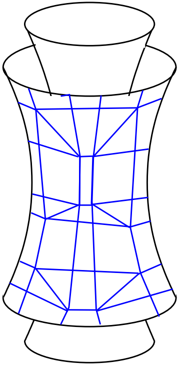

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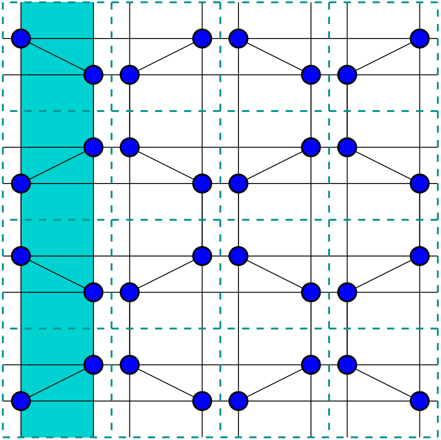

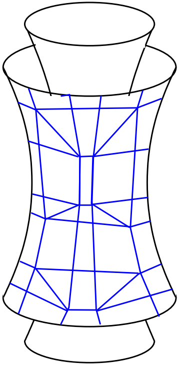

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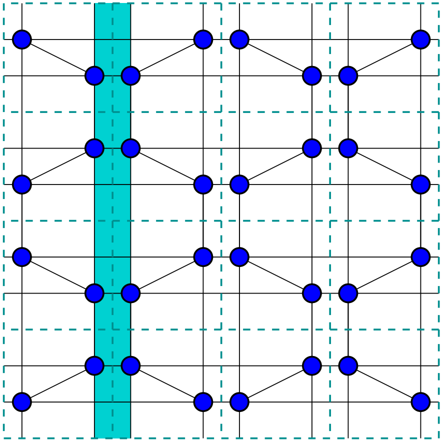

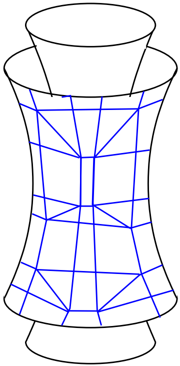

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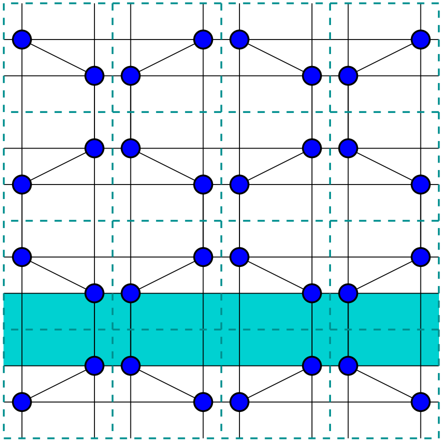

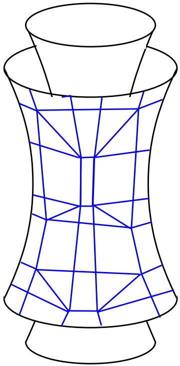

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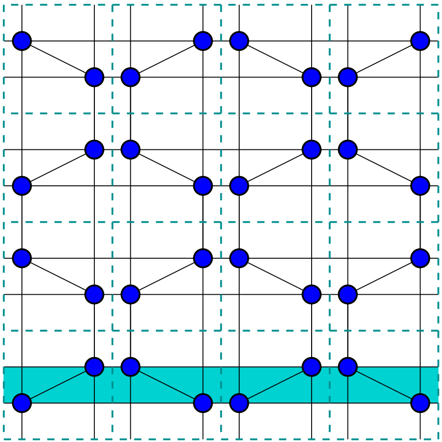

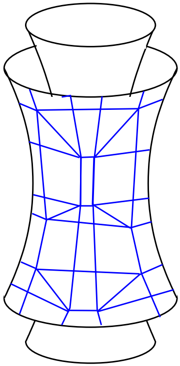

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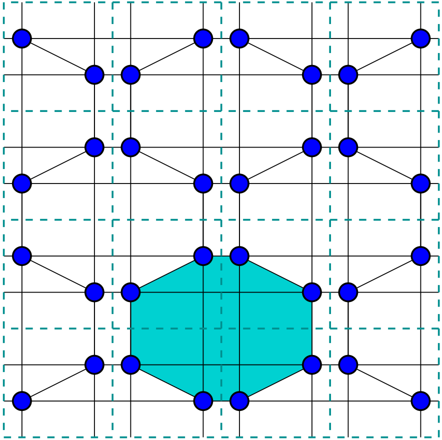

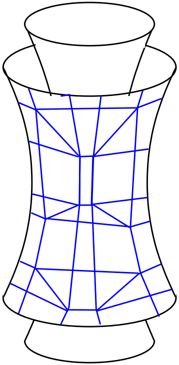

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The following prismatoid Q, of dimension 5 and with 48 vertices, has width six.

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$$Q := \operatorname{conv}\left\{ \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ \pm 18 & 0 & 0 & 0 & 1 \\ 0 & \pm 18 & 0 & 0 & 1 \\ 0 & 0 & \pm 45 & 0 & 1 \\ 0 & 0 & 0 & \pm 45 & 1 \\ \pm 15 & \pm 15 & 0 & 0 & 1 \\ 0 & 0 & \pm 30 & \pm 30 & 1 \\ 0 & \pm 10 & \pm 40 & 0 & 1 \\ \pm 10 & 0 & 0 & \pm 40 & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & 0 & \pm 18 & -1 \\ 0 & 0 & \pm 18 & 0 & -1 \\ \pm 45 & 0 & 0 & 0 & -1 \\ 0 & \pm 45 & 0 & 0 & -1 \\ 0 & 0 & \pm 15 & \pm 15 & -1 \\ \pm 30 & \pm 30 & 0 & 0 & -1 \\ \pm 40 & 0 & \pm 10 & 0 & -1 \\ 0 & \pm 40 & 0 & \pm 10 & -1 \end{pmatrix} \right\},$$

A 5-prismatoid of width > 5

Theorem

The following prismatoid Q, of dimension 5 and with 48 vertices, has width six.

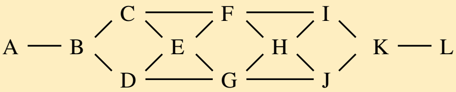
Corollary

There is a 43-dimensional polytope with 86 facets and diameter (at least) 44.

A 5-prismatoid of width > 5

Proof 1.

It has been verified computationally that the dual graph of Q (modulo symmetry) has the following structure:

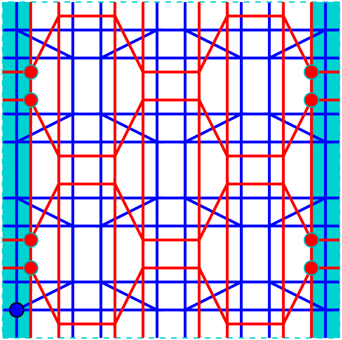


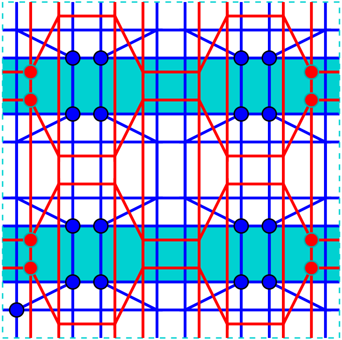
□

A 5-prismatoid of width > 5

Proof 2.

Show that there are no blue vertex a and red vertex b such that a is a vertex of the blue cell containing b and b is a vertex of the red cell containing a. □





Smaller 5-prismatoids of width > 5

With the same ideas

Theorem

The following 5-prismatoid with 28 vertices (and 274 facets) has width 6.

$$Q := \mathrm{conv}\left\{ \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ \pm 18 & 0 & 0 & 0 & 1 \\ 0 & 0 & \pm 30 & 0 & 1 \\ 0 & 0 & 0 & \pm 30 & 1 \\ 0 & \pm 5 & 0 & \pm 25 & 1 \\ 0 & 0 & \pm 18 & \pm 18 & 1 \end{pmatrix} \quad \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & \pm 18 & 0 & -1 \\ 0 & \pm 30 & 0 & 0 & -1 \\ \pm 30 & 0 & 0 & 0 & -1 \\ \pm 25 & 0 & 0 & \pm 5 & -1 \\ \pm 18 & \pm 18 & 0 & 0 & -1 \end{pmatrix} \right\},$$

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There is a non-Hirsch polytope of dimension 23 with 46 facets.

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And with some more work:

Theorem (Matschke-Santos-Weibel, 2011)

There is a 5-prismatoid with 25 vertices and of width 6.

Corollary

There is a non-Hirsch polytope of dimension 20 with 40 facets.

This one has been explicitly computed. It has $36,442$ vertices, and diameter 21.

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poly20dim21.ext

```
V-representation
begin
 40 21 rational
1  1        0      0    21      0         0         0         0           0            0            0            0            0      0      0      1      1      0      0      0
1  1        0      0    16     -8         0         0         0           0            0            0            0            0      0      0      0      1      1      0      0
1  1        0      0     0    -25         0         0         0           0            0            0            0            0      0      0      0      0      1      1      0
1  1        0      0   -16    -10         0         0         0           0            0            0            0            0      0      0      0      0      0      1      1
1  1        0      0   -21      0         0         0         0           0            0            0            0            0      0      0      0      0      0      0      1
1  1        0      0   -20      3         0         0         0           0            0            0            0            0      0      0      0      0      0      0      0
1  1        0      0     0     39         0         0         0           0            0            0            0            0      0      0      0      0      0      0      0
1  1        0      0    19      7         0         0         0           0            0            0            0            0      0      0      0      0      0      0      0
1  1    3/100  -1/50     0    -23         0         0         0           0            0            0            0            0      0      0      0      0      0      0      0
1  1   -3/100  -1/50     0     37         0         0         0           0            0            0            0            0      0      0      0      0      0      0      0
1  1  -3/2000 7/2000     0 388/10         0         0         0           0            0            0            0            0      0      1      0      0      0      0      0
1  1   3/2000 7/2000     0 -248/10  10000000  10000000  10000000  10000000000  100000000000  100000000000  100000000000  100000000000      1      0      0      0      0      0      0
1  1   3/2000 7/2000     0 -248/10 -10000000         0         0           0            0            0            0            0      1      0      0      0      0      0      0
1  1   3/2000 7/2000     0 -248/10  10000000 -10000000         0           0            0            0            0            0      1      0      0      0      0      0      0
1  1   3/2000 7/2000     0 -248/10  10000000  10000000 -10000000           0            0            0            0            0      1      0      0      0      0      0      0
1  1   3/2000 7/2000     0 -248/10  10000000  10000000  10000000 -10000000000            0            0            0            0      1      0      0      0      0      0      0
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1  1   3/2000 7/2000     0 -248/10  10000000  10000000  10000000  10000000000  100000000000 -100000000000            0            0      1      0      0      0      0      0      0
1  1   3/2000 7/2000     0 -248/10  10000000  10000000  10000000  10000000000  100000000000  100000000000 -100000000000            0      1      0      0      0      0      0      0
1  1   3/2000 7/2000     0 -248/10  10000000  10000000  10000000  10000000000  100000000000  100000000000  100000000000 -100000000000      1      0      0      0      0      0      0
1 -1       30      0     0      0         0         0         1           1            0            0            0            0      0      0      0      0      0      0      0
1 -1        4    -15     0      0         0         0         0           1            1            0            0            0      0      0      0      0      0      0      0
1 -1        0  -33/2     0      0         0         0         0           0            1            1            0            0      0      0      0      0      0      0      0
1 -1       -1    -16     0      0         0         0         0           0            0            1            1            0      0      0      0      0      0      0      0
1 -1    -55/2      0     0      0         0         0         0           0            0            0            1            1      0      0      0      0      0      0      0
1 -1      -17     18     0      0         0         0         0           0            0            0            0            1      0      0      0      0      0      0      0
1 -1        0     38     0      0         0         0         0           0            0            0            0            0      0      0      0      0      0      0      0
1 -1       22     17     0      0         0         0         0           0            0            0            0            0      0      0      0      0      0      0      0
1 -1      -10      0   1/5   -1/5         0         0         0           0            0            0            0            0      0      0      0      0      0      0      0
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1 -1 299999/10000  0     0  1/100         0         1         0           0            0            0            0            0      0      0      0      0      0      0      0
1 -1 -2745/100     0 1/5000 1/800         1         0         0           0            0            0            0            0      0      0      0      0      0      0      0
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1 -1      -27      0 1/500  -1/80         0         0         0           0            0            0            0            0 -100000         0         0         0          0          0           0
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1 -1      -27      0 1/500  -1/80         0         0         0           0            0            0            0            0  100000  10000000 -10000000         0          0          0           0
1 -1      -27      0 1/500  -1/80         0         0         0           0            0            0            0            0  100000  10000000  10000000 -10000000          0          0           0
1 -1      -27      0 1/500  -1/80         0         0         0           0            0            0            0            0  100000  10000000  10000000  10000000 -100000000          0           0
1 -1      -27      0 1/500  -1/80         0         0         0           0            0            0            0            0  100000  10000000  10000000  10000000  100000000 -100000000           0
1 -1      -27      0 1/500  -1/80         0         0         0           0            0            0            0            0  100000  10000000  10000000  10000000  100000000  100000000 -1000000000
end
allbases
printcobasis
```

Asymptotic width in dimension five

Theorem

There are 5-dimensional prismatoids with n vertices and width $\Omega(\sqrt{n})$.

Sketch of proof

Start with the following “simple, yet more drastic” pair of maps in the torus.

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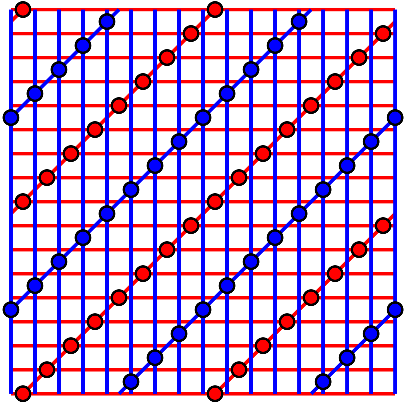

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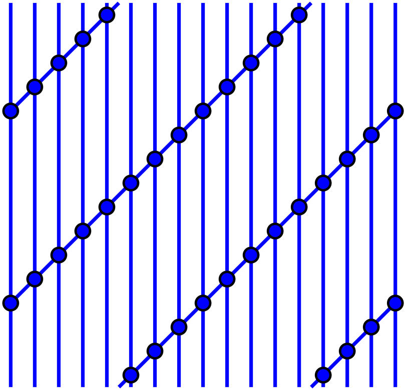

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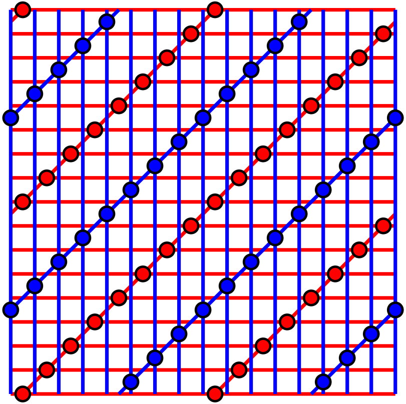

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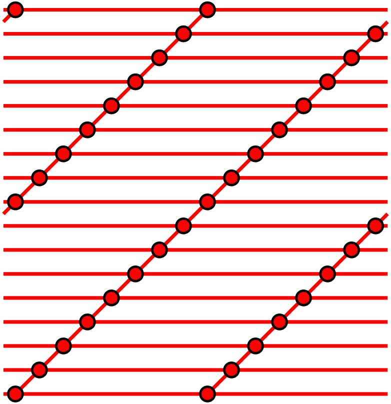
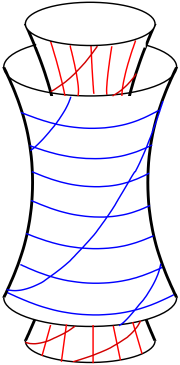


Asymptotic width in dimension five

Consider the red and blue maps as lying in two parallel tori in the 3-sphere.



Complete the tori maps to the whole 3-sphere (you need quadratically many cells for that).

Between the two tori you basically get the superposition of the two tori maps. □

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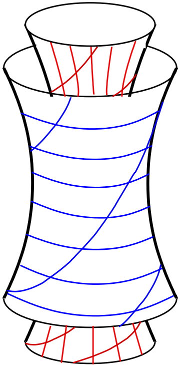

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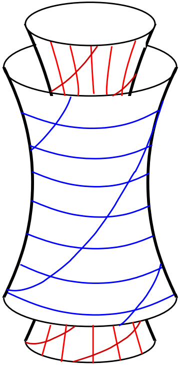

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Hirsch Wars Episode III
Revenge of the linear bound

Francisco Santos
http://personales.unican.es/santosf

Departamento de Matemáticas, Estadística y Computación
Universidad de Cantabria, Spain

MDS Summer Schhol, Döllnsee — August 14–16, 2012

Previously on Hirsch wars. . .

The Phantom Conjecture

Let $H(n,d) := \max\{\text{diam}(P) : P \text{ is a } d \text{ polytope with } n \text{ facets}\}$.

Conjecture: Warren M. Hirsch (1957)

$$\forall n,d, \qquad H(n,d) \leq n-d.$$

Theorem [Kalai-Kleitman 1992]

$$H(n,d) \leq n^{\log_2 d+2}, \qquad \forall n,d.$$

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$$H(n,d) \leq n2^{d-3}, \qquad \forall n,d.$$

Polynomial Hirsch conjecture/question

There are c and k such that $H(n,d) \leq c \cdot (n-d)^k$, $\forall n,d$

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Let $H(n,d) := \max\{\text{diam}(P) : P \text{ is a } d \text{ polytope with } n \text{ facets}\}$.

Conjecture: Warren M. Hirsch (1957)

$$\forall n, d, \qquad H(n,d) \leq n - d.$$

Theorem [Kalai-Kleitman 1992]

$$H(n,d) \leq n^{\log_2 d + 2}, \qquad \forall n, d.$$

Theorem [Barnette 1967, Larman 1970]

$$H(n,d) \leq n 2^{d-3}, \qquad \forall n, d.$$

Polynomial Hirsch conjecture/question

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Attack of the Prismatoids

Theorem (Strong d-step Theorem, S. 2010)

If a prismatoid Q has width $> d$ then there is another prismatoid Q' (of dimension $n-d$, with $2n-2d$ facets, and width $\geq \delta + n - 2d > n-d$) that violates (the dual of) the Hirsch conjecture.

Theorem (Matschke-S.-Weibel 2012)

There is a 5-dim. prismatoid of width 6 with 25 vertices.

Corollary

There is a non-Hirsch polytope of dimension 20 with 40 facets.

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Once we have a non-Hirsch polytope we can derive more via:

1. Products of several copies of it (dimension increases).
2. Gluing several copies of it (dimension is fixed).

To analyze the asymptotics of these operations, we call excess of a d-polytope P with n facets and diameter δ the number

$$\epsilon(P) := \frac{\delta}{n-d} - 1 = \frac{\delta - (n-d)}{n-d}.$$

E. g.: The excess of our non-Hirsch polytope with $n - d = 20$ and with diameter 21 is

$$\frac{21-20}{20} = 5\%.$$

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1. Taking products preserves the excess: for each $k \in \mathbb{N}$, there is a non-Hirsch polytope of dimension $20k$ with $40k$ facets and with excess equal to $0.05 = 5\%$.
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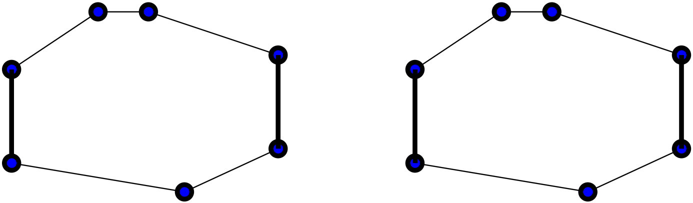

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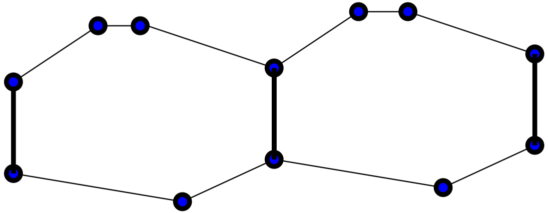

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$$\delta = \delta_1 + \delta_2 - 1$$

$$\frac{\delta_1}{n_1 - d} - 1 = \frac{\delta_2}{n_2 - d} - 1 = \epsilon \quad \Rightarrow \quad \frac{\delta}{n - d} - 1 = \epsilon - \frac{1}{(n_1 - d) + (n_2 - d)}.$$

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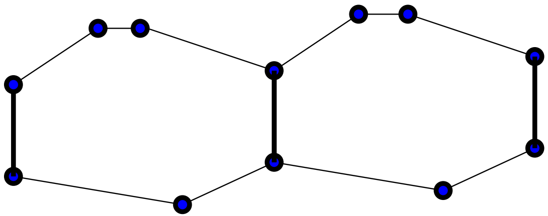

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Corollary

For each $k \in \mathbb{N}$ there is an infinite family of non-Hirsch polytopes of fixed dimension $20k$ and with excess (tending to)

$$0.05\left(1 - \frac{1}{k}\right).$$

The excess of a prismatoid

But we know there are "worst" prismatoids: 5-prismatoids of arbitrarily large width. Will those produce non-Hirsch polytopes with worst excess?

To analyze the asymptotics of this, let us call *excess* of a prismatoid of width δ with n vertices and dimension d the quantity

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Lemma

Via the strong d-step Theorem, a prismatoid of a certain excess produces non-Hirsch polytopes of that same excess.

Proof.

The dimension, number of facets and diameter of the non-Hirsch polytope produced by the strong d-step Theorem are

$$n-d, \qquad 2(n-d), \qquad \delta+(n-2d).$$

So, its excess is

$$\frac{\delta+(n-2d)-(n-d)}{n-d}=\frac{\delta-d}{n-d}.$$

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Prismatoids of large width won't help (much)

In dimension 5, we know how to construct polytopes of arbitrarily large width $\delta \sim \sqrt(n)$... but their excess tends to zero:

$$\lim \frac{\delta - 5}{n - 5} = \lim \frac{\sqrt{n} - 5}{n - 5} = 0.$$

Let us be optimistic and suppose that we could construct 5-prismatoids with n vertices and linear width $\simeq \alpha n$.

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Revenge of the linear bound

OK, let us be *more* optimistic. Can we hope for prismatoids of width greater than linear?

In fixed dimension, certainly not:

Theorem

The width of a d-dimensional prismatoid with n vertices cannot exceed $2^{d-3}n$.

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This is a general result for the (dual) diameter of a polytope [Barnette, Larman, ∼1970]. □

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Corollary

Using the Strong d-step Theorem for 5-prismatoids it is impossible to violate the Hirsch conjecture by more than 33%.

Thank you

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