MDS Summer School "Combinatorics of Linear and Semidefinite Programming"

Diameter of polytopes and the Hirsch Conjecture, Problem sheet.

1. **Diameter of 3-polytopes** (Klee 1966, for parts a, b and c).
   a) Prove the (bounded) Hirsch Conjecture in dimension 3. More precisely, show the following, where $H(d,n)$ denotes the maximum diameter among d-polytopes with n facets:
   $$H(3,n) \le \left\lfloor \frac{2n}{3} \right\rfloor - 1.$$
   Hint: use Euler's formula and the fact that d-polytope graphs are d-connected. Q: why does this not tell you anything (useful) about $H(4,n)$?
   b) Show that the upper bound in the previous exercise is exact.
   Hint: do this by an explicit construction. It may be easier to think in the dual setting where you have a simplicial 3-polytope and move from facet to neighboring facet rather than vertex to neighboring vertex.
   c) More generally, show that for every n and d
   $$H(d,n) \ge \left\lfloor \frac{(d-1)n}{d} \right\rfloor - (d-2).$$
   Hint: glue $\simeq n/d$ cross-polytopes of dimension d to one another. A cross-polytope is the dual of a d-cube. "Glue" means that you identify one facet of each copy with one of the next one; you need to show that that can be done in a way that gives a *convex* polytope, for which you are allowed to geometrically transform your polytope into a combinatorially isomorphic one
   d) Let us call a simplicial 3-polytope *flag* if the only 3-cycles in its graph are those corresponding to triangular faces (e.g.: a 3-polytope with a vertex of degree 3 is not flag, unless it is a tetrahedron). Let P be a simple 3-polytope with a flag dual. Show that its diameter is at most $\frac{n}{2}$. Think of a 3-polytope that achieves the bound.
   Hint: use that the graph of P is "almost" 4-connected (and make sense out of this sentence. The graph cannot be 4-connected since every vertex has degree three).
2. **Hisch-sharp polyhedra, part 1**
   a) Show that for every d and n there is a *Hirsch sharp* unbounded d-polyhedron with n facets (that is, a polyhedron whose diameter meets the Hirsch bound with equality).
   b) Show that for every d and n satisfying $d < n \le 2d$ there is a *Hirsch sharp* polytope.
3. **Iterated wedge** We define the k-th iterated wedge of a polytope P at a facet F as follows: if $k = 1$ this is just the usual wedge. If $k > 1$ then we first wedge P at F to get a new polytope P' and then perform a $(k-1)$-th wedge on P' at one of the two special facets produced by the wedge at P (we call "special" the two facets that are "copies of P").

   Let u and v be vertices of a polytope P at a certain distance δ, and let P_k be the k-th iterated wedge of P at a facet F not containing neither u nor v. Show that:
   a) P_k has $n+k$ facets and $d+k$ dimensions, where n and d are those of P.

b) P_k contains $k+1$ vertices $u_0, \dots u_k$ "coming from u", that induce a clique (=complete subgraph) in the graph of P_k. Same for v.
c) Any u_i is at distance at least δ from any v_j.
d) Moreover, every u_i and every v_j are each incident to k edges with the property that any path from u_i to v_j that uses any of those edges has length *strictly* larger than δ.

4. **Hisch-sharp polytopes, part 2** (Fritzsche-Holt 1999) Let P be a Hirsch-sharp simple d-polytope with n facets. Let u and v be vertices at distance $n-d$. Let P' be the d-th iterated wedge of P at a facet F not containing neither u nor v.
   a) Show that arbitrarily many copies of P' can be "glued" to one another to get Hirsch-sharp polytopes of dimension $2d$ with arbitrarily many facets. Hint: if you did 1.c then you are almost there.
   b) Conclude that in any dimension $d \geq 8$ there are arbitrarily large Hirsch-sharp polytopes. (Note: this is also known to hold in $d = 7$, but there the construction is a bit more subtle).

5. **Excess of a polytope** A non-Hirsch polytope is a polytope with diameter exceeding the Hirsch bound. Define the *excess* of a non-Hirsch d-polytope P with n facets to be the quantity

$$\frac{\mathrm{diam}(P)}{n-d} - 1.$$

   a) Show that the excess of a polytope is preserved by taking Cartesian products with itself.
   b) Relate the excess of two poytopes P and Q with that of the "gluing" of them used in the last exercise. In particular, show that gluing several copies of P gives a polytope with excess strictly smaller than the original one.

6. **Monotone and unbounded Hirsch conjectures** Complete/repeat the proof of the equivalence of the following statements:
   a) There is a Hirsch-sharp 4-polytope with 9 facets (Klee-Walkup 1967).
   b) There is a non-Hirsch unbounded 4 polyhedron with 8 facets (Klee-Walkup 1967).
   c) There is a 4-polytope P with 8 facets in which the "monotone Hirsch conjecture" fails. That is, there is a certain linear functional f and a certain initial vertex u such that no f-monotone path goes from u to the f-maximal vertex v in 4 steps or less. (Todd 1980)

   Remark: the numbers $d = 4$ and $n = 9$ are irrelevant. But the fact that $n > 2d$ *is* relevant.

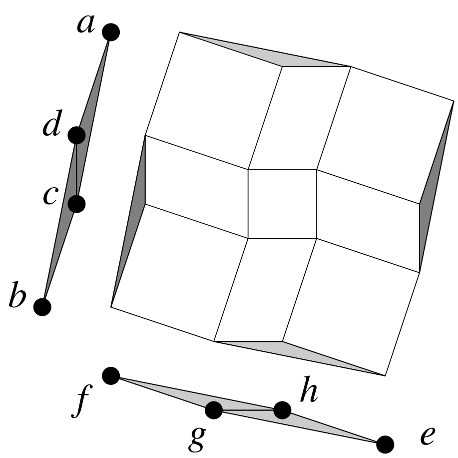
7. **Excess of a polytope, part II** Repeat/complete the proof of the following:

   Lemma: Knowing the existence of a 20-polytope with 40 facets and diameter 21, show that for every (fixed) $k \in \mathbb{N}$ there is an infinite family of polytopes of dimension $20k$ and with excess limiting to (but strictly smaller than)

$$0.05\left(1 - \frac{1}{k}\right).$$

8. **The Klee-Walkup "unbounded non-Hirsch 4-spindle"** Interpret the Klee-Walkup non-Hirsch 4-polyhedron with 8 facets as an "unbounded 4-spindle". What is the associated "pair of maps on a surface"? Check that you cannot go from a blue vertex to a red vertex in 2 steps.

   Hint: stare closely at this picture until you see a prismatoid in it:



Exercises for Episode IV (Not covered in the course)

9. A connected layer family is called *complete* if its underlying simplicial complex contains all the possible simplices of that rank and number of vertices. It is called *injective* if each layer has a single simplex. Show that the length of a complete connected layer family of rank d with n vertices cannot be greater than: $(n-d)d$ and that of an injective one cannot be greater than $n-d$.

   Hints: for the injective case, observe that from one layer to the next some symbol must appear, and no symbol can appear at two different steps. For the complete case let u and v be the first and last simplex, and let w be a simplex from an intermediate layer and with $w \subset u \cup v$. Then apply induction to the links of $w \cap u$ and $w \cap v$.

10. Repeat the previous exercise for multifamilies, to obtain the bound $(n-d)d$ both in the complete and in the injective case.