An invariant effective description of compact non-convex polyhedra

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It is often assumed that the only way to represent a (compact) non-convex polyhedron $X \subset \mathbb{R}^d$ is by partitioning into convex polytopal pieces. Such partitions are very far from being unique or succinct; in particular in dimensions $d > 2$ one might need to introduce *Steiner points*—points in the inerior of X which must arise as vertices of polytopal pieces.

We propose a description that is essentially unique: a rational function associated with the uniform measure $\mu(X)$ supported on X. It is given by an integral transform of $\mu(X)$ known as *Fantappiè transformation*

$$\mathcal{F}(\mu(X))(u) := d! \int_{\mathbb{R}^d} \frac{d\mu(X)(x)}{(1 - \langle u, x \rangle)^{d+1}} = \frac{P(u)}{\prod_{v \in V}(1 - \langle u, v \rangle)},$$

where $\langle x, y \rangle$ denotes the standard scalar product $\sum_i x_i y_i$. Expanding $\mathcal{F}$ into Taylor series gives a scaled moment generating function (m.g.f.) for $\mu(X)$, known in the univariate case as factorial m.g.f., and this allows for efficient computation of integrals over X, etc.

$\mathcal{F}$ admits decompositions into "elementary" summands with real weights, corresponding to certain simplices with vertices from V, which are analogous to triangulations of convex polytopes. In particular they allow an efficient way to check containment of a point in X. It is an interesting open question whether these weights can be chosen to be ± 1.

It is natural to homogenise $\mathcal{F}$, an operation that corresponds to switching to the Laplace-Fourier transform of a measure supported on the conic closure of X embedded into the hyperplane $\{x_0 = 1\} \subset \mathbb{R}^{d+1}$. It allows to apply powerful machinery of Jeffrey-Kirwan residues by Brion and Vergne, used in the theory of hyperplane arrangements.