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Functional Analysis I

1st problem sheet

Please return your responses in the tutorials on April, 24th / 25th.

Problem 1: **4 pt.**

Show that $(l^\infty(\mathbb{N}), \|\cdot\|_\infty)$ with $\|(t_n)\|_\infty := \sup_{n\in\mathbb{N}} |t_n|$ is a Banach space.

Problem 2: **4 pt.**

Let $\mathcal{P}$ be the vector space of all polynomials with real coefficiants, i.e.

$$\mathcal{P} := \Big\{p : \mathbb{R} \to \mathbb{R} \mid p(x) = \sum_{i=0}^{n} a_i x^i,\ a_i \in \mathbb{R}, i = 0, \ldots, n;\ n \in \mathbb{N}\Big\}.$$

Which of the following mappings defines a norm on $\mathcal{P}$?

$$(i)\ p \mapsto |a_0|, \quad (ii)\ p \mapsto |a_n|, \quad (iii)\ p \mapsto |a_n| + \cdots + |a_0|.$$

Problem 3: **5 pt.**

Let (X, d) be a metric space and define T as follows

$$T := \{A \subset X \mid A \text{ is nonempty, closed and bounded}\}.$$

For $\epsilon > 0$ and $A \in T$ denote by $B_\epsilon(A)$ the ϵ-neighbourhood of A, i.e.

$$B_\epsilon(A) = \{x \in X \mid d(x, A) = \inf_{y\in A} d(x, y) < \epsilon\}.$$

Show that

$$d_H(A_1, A_2) := \inf\{\epsilon > 0 \mid A_1 \subset B_\epsilon(A_2) \text{ and } A_2 \subset B_\epsilon(A_1)\},\ A_1, A_2 \in T$$

defines a metric on T.

Problem 4: **7 pt.**

Let $A \subset \mathbb{R}^n$ and $x_0 \in A$. For $\lambda \in (0, 1]$ define the class $C^\lambda(A)$ of Hölder continuous functions (cf. exercise) via

$$C^\lambda(A) := \{f : A \to \mathbb{R} \mid |f|_\lambda < \infty\},$$
$$|f|_\lambda := \sup_{x,y\in A, |x-y|<1} \frac{|f(x) - f(y)|}{|x - y|^\lambda},$$

and the class of bounded continuous functions

$$C_b(A) := \{f : A \to \mathbb{R} \mid f \text{ is continuous and } \|f\|_\infty < \infty\}.$$

(i) Let $\lambda \in (0,1]$ and set

$$M := \{f \in C^\lambda(A) \cap C_b(A) \mid |f|_\lambda \leq 1\}.$$

(ii) Define

$$N := \{f \in C(A) \mid f(x) \in [0,1) \text{ for all } x \in A\}.$$

Are the sets M or N open or closed subsets of $C_b(A)$ (equipped with the maximum norm)?
Furthermore show $C^\lambda(A) \subset C_b(A)$ if A is bounded.