TECHNISCHE UNIVERSITÄT BERLIN
INSTITUT FÜR MATHEMATIK
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Functional Analysis I

10th problem sheet

Please return your answers in the tutorials on June, 26th / 27th.

Problem 1: **5 pt.**

Let $1 < p < \infty$, $a = (a_1, a_2, \dots) \in l^\infty(\mathbb{N})$, $N \in \mathbb{N}$ and define $T_i : l^p(\mathbb{N}) \to l^p(\mathbb{N})$, $i = 1, 2, 3$ by

$$T_1 x := (a_1 x_1, a_2 x_2, \dots),$$
$$T_2 x := (0, 0, a_1 x_1, a_2 x_2, \dots),$$
$$T_3 x := (x_1, x_2, \dots, x_N, 0, \dots),$$

for $x \in l^p(\mathbb{N})$. Show that these operators are bounded and determine the corresponding adjoint operators.

Problem 2: **5 pt.**

Let X, Y and Z be Banach spaces and let $S \in L(X, Z)$, $T \in L(Y, Z)$. Assume that for every $x \in X$ there is an unique $y \in Y$ with $Sx = Ty$. Define $A : X \to Y$ to be the operator that maps $x \in X$ to this unique $y \in Y$. Show that A is linear and bounded.

Problem 3: **6 pt.**

Let $g : \mathbb{R} \to \mathbb{R}$ be measurable. We define

$$\mathcal{D} := \{ f \in L^2(\mathbb{R}) \mid gf \in L^2(\mathbb{R}) \} \subset L^2(\mathbb{R})$$

and

$$T : \mathcal{D} \to L^2(\mathbb{R}), \quad f \mapsto Tf := gf.$$

(i) Show that $\|(T - \lambda) f\|_2 \geq |\mathrm{Im}\lambda| \|f\|_2$ for all $f \in \mathcal{D}$, $\lambda \in \mathbb{C}$[1].

(ii) Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Show that the operator $T - \lambda$ is bijective and the inverse $(T - \lambda)^{-1}$ is bounded. Calculate the inverse $(T - \lambda)^{-1}$ in this case. Show that both $T - \lambda$ and T are closed operators.

[1] $\mathrm{Im}\lambda$ is the imaginary part of λ.

Problem 4: **4 pt.**

(i) Give a counterexample that shows that the completeness of X is necessary in the closed graph theorem.

(ii) Give a counterexample that shows that the completeness of Y is necessary in the closed graph theorem.