TECHNISCHE UNIVERSITÄT BERLIN
INSTITUT FÜR MATHEMATIK
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Functional Analysis I

12th problem sheet

This is the last problem sheet.Please return your answers in the tutorials on July, 10th / 11th.

Problem 1: **5 pt.**
Let $(H, (\cdot, \cdot)$ be a Hilbert space and $U, V \subset H$ be closed subspaces. Let P_U and P_V the corresponding orthogonal projections. Show

(i) $U \subset V$ if and only if $P_V P_U = P_U P_V = P_U$.

(ii) $U \perp V$ if and only if $P_U P_V = 0$.

(iii) $P_U P_V$ is an orthogonal projection (onto which subspace?) if and only if $P_U P_V = P_V P_U$.

Problem 2: **8 pt.**
Let $(H, (\cdot, \cdot)$ be a Hilbert space, $f(z) := \sum_{n=0}^{\infty} a_n z^n$ a power series with radius of convergence $R \in (0, \infty]$. Let $A \in L(H)$ with $\|A\| < R$. Show that there exists a uniquely defined operator $T \in L(H)$ with

$$(v, Tu) = \sum_{n=0}^{\infty} a_n (v, A^n u), \quad u, v \in H.$$

Furthermore show

$$\|T - \sum_{n=0}^{N} \alpha_n A^n\| \to 0 \quad \text{as } N \to \infty.$$

Remark: One often notates $T = f(A)$.

Problem 3: **3 pt.**
Find a normed space $(X, \|\cdot\|)$ such that the norm $\|\cdot\|$ is not induced by an inner product.

Problem 4: **4 pt.**
Let H be a vector space equipped with an inner product $(\cdot, \cdot)$. Show that two elements $u, v \in H$ are orthogonal if and only if $\|u + \lambda v\| = \|u - \lambda v\|$ for all $\lambda \in \mathbb{K}$.