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Summer semester 2008

Functional Analysis I

4th problem sheet

Please return your answers in the tutorials on May, 15th / 16th.

Problem 1: **4 pt.**

Let $(X, \|\cdot\|)$ be a normed space and let $Y \subset X$ be a closed subspace. Show that X is separable if and only if Y and X/Y are separable.

Problem 2: **6 pt.**

(i) Let $1 \le p, q \le \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$ (with the usual convention $\frac{1}{\infty} = 0$). Fix $y \in l^q(\mathbb{N})$. Calculate the operator norm of the functional (i.e. $\mathbb{K}$-valued operator)

$$T : l^p(\mathbb{N}) \to \mathbb{R}, \quad x \mapsto Tx := \sum_{i=1}^{\infty} x_n y_n.$$

(ii) Let $n \in \mathbb{N}$, $t_1, \dots, t_n \in [0,1]$, $t_i \neq t_j$ if $i \neq j$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. Calculate the operator norm of the functional

$$S : C([0,1]) \to \mathbb{R}, \quad f \mapsto Sf := \sum_{i=1}^{n} \alpha_i f(t_i).$$

Here all spaces are equipped with their natural norms.

Problem 3: **4 pt.**

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and $T : X \to Y$ be a continuous, linear operator. The operator norm $\|T\|$ of T is defined by

$$\|T\| := \inf\{M > 0 \,|\, \|Tx\|_Y \le M\|x\|_X \text{ for all } x \in X\},$$

Show that

$$\|T\| = \sup_{x \in X, x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{x \in X, \|x\|_X = 1} \|Tx\|_Y = \sup_{x \in X, \|x\|_X \le 1} \|Tx\|_Y$$

and

$$\|Tx\|_Y \le \|T\| \|x\|_X \text{ for all } x \in X$$

hold.

Problem 4: **6 pt.**

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Recall that an operator $T : X \to Y$ is called *bounded* if A maps bounded sets into bounded sets. We know that *linear* operators are bounded if and only if they are continuous. This is not the case for nonlinear operators:

Let $X = Y = l^2(\mathbb{N})$ equipped with the $\|\cdot\|_2$−norm. Let

$$\overline{B}(0,1) := \{u \in l^2(\mathbb{N} \mid \|u\|_2 \leq 1\}$$

and consider the mapping

$$T : \overline{B}(0,1) \to l^2(\mathbb{N}), \quad u \mapsto Tu := \Big(\sum_{k=1}^{\infty} \frac{2^{-k}}{1 + 3^{-k} - u_k}, 0, 0, \dots\Big).$$

Show that T is well defined and continuous, but T maps $\overline{B}(0,1)$ into an unbounded set.