TECHNISCHE UNIVERSITÄT BERLIN
INSTITUT FÜR MATHEMATIK
J. Behrndt, C. Kreusler

Summer semester 2008

Functional Analysis I

5th problem sheet

Please return your answers in the tutorials on April, 22nd / 23rd.

Problem 1: **4 pt.**

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and $T : X \to Y$ linear and bounded, i.e. $T \in L(X, Y)$.

(i) Are the kernel $\ker T$ and the range $\operatorname{ran} T$ linear subspaces of X, Y resp. ? Are they closed?

(ii) Does always exist an $x \in X$, $x \neq 0$, such that $\|Tx\| = \|T\|\|x\|$?

Problem 2: **5 pt.**

Let $G : [0, \pi] \times [0, \pi] \to \mathbb{R}$ be defined by

$$G(x,y) := \begin{cases} \sin(\frac{1}{2}(x-\pi))\sin(\frac{1}{2}y) & \text{for } x > y, \\ \sin(\frac{1}{2}(y-\pi))\sin(\frac{1}{2}x) & \text{for } x \leq y. \end{cases}$$

Furthermore define

$$T : L^2([0,\pi]) \to L^2([0,\pi]), \quad u \mapsto (Tf)(x) := 2\int_0^\pi G(x,y)f(y)dy.$$

(i) Show that T is a well defined, linear and bounded operator.

(ii) Let $f : [0, \pi] \to \mathbb{R}$ be continuous. Show that Gf solves the boundary value problem (BVP)

$$\begin{cases} u''(x) + \frac{1}{4}u'(x) = f(x), & x \in (0,\pi), \\ u(0) = u(\pi) = 0. \end{cases}$$

Problem 3: **5 pt.**

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix and let

$$r(A) := \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } A\}$$

be the spectral radius of A. We equip $\mathbb{R}^n$ with the usual euclidean norm and consider the corresponding induced operator norm $\|A\|$.

Show $\|A\| = r(A)$ and prove that $r(A) < 1$ holds if and only if $\|A^n\| \to 0$ as $n \to \infty$.

Problem 4: **6 pt.**

Let $(X \cdot)$ be a real, strictly convex normed space and let $\Phi : X \to X$ be a surjective operator with $\Phi(0) = 0$ and

$$\|\Phi(x) - \Phi(y)\| = \|x - y\| \quad \text{for all } x, y \in X.$$

Show that Φ is linear.

Hint: First show that $z = \frac{1}{2}(x + y)$ is the only element in X such that $\|z - x\| = \|z - y\| = \frac{1}{2}\|x - y\|$ holds.

Remark: The statement is also true if X is not strictly convex, but much harder to prove then.

Bonus problem 1: **5 Bonus pt.**

Let $(X, \|\cdot\|)$ be a normed space and let $S, T : X \to X$ be linear operators such that

$$ST - TS = Id_X$$

holds. Show that S or T has to to be unbounded.

Hint: First show $ST^{n+1} - T^{n+1}S = (n+1)T^n$ and assume that S and T are bounded.

Remark: If $X = C^\infty(\mathbb{R})$ equipped with some norm and $(Sf)(t) := f'(t)$, $(Tf)(t) = tf(t)$ then one may directly calculate that $ST - TS = Id$ holds. In fact this is the one-dimensional analogon of the Heisenberg uncertainty principle in quantum mechanics. S represents the momentum (Impuls) and T the position. In quantum mechanics the operators of interest are unbounded!