Technische Universität Berlin
Institut für Mathematik
J. Behrndt, C. Kreusler

Summer semester 2008

Functional Analysis I

8th problem sheet

Please return your answers in the tutorials on June, 12th / 13th.

Problem 1: **4 pt.**

Let $(X, \|\cdot\|)$ be a normed space and let U be a subspace of X. Show that

$$\overline{U} = \bigcap_{f \in X',\, U \subset \ker f} \ker f.$$

Problem 2: **5 pt.**

Let $(X, \|\cdot\|)$ be a separable Banach space. Show that there exists a linear isometric Operator $T : X \to l^\infty(\mathbb{N})$. Is X isometric isomorph to a closed subspace of $l^\infty(\mathbb{N})$?

Problem 3: **5 pt.**

A linear mapping $T : C([a,b]) \to \mathbb{R}$ is a *positive* functional on $C([a,b])$ if $Tf \geq 0$ holds for all $f \in C([a,b])$ with $f \geq 0$. For other spaces of functions or sequences this definition shall be suitable adapted.

(i) Show that every positive functional T on $C([a,b])$ is continuous and calculate the norm of T.

(ii) Let $S : L^\infty([a,b]) \to \mathbb{R}$ be an extension of T with $\|S\| = \|T\|$. Show that S is a positive functional on $L^\infty([a,b])$.

Problem 4: **6 pt.**

Consider $X = l^\infty(\mathbb{N})$ equipped with the norm $\|\cdot\|_\infty$. Show that there is a linear functional $F : X \to \mathbb{R}$ with the following properties:

(i) F is a positive functional on $l^\infty(\mathbb{N})$.

(ii) Let $S : X \to X$ be the shift operator defined by

$$x = (x_1, x_2, \dots) \mapsto Sx := (x_2, x_3, \dots).$$

Then $F(Sx) = Fx$ for all $x \in X$.

(iii) $F\mathbf{1} = 1$, where $\mathbf{1} := (1, 1, 1, \dots)$.

Furthermore, show that F is continuous with norm $\|F\| = 1$ and that for converging sequences $x = (x_n) \in l^\infty(\mathbb{N})$, we have $Fx = \lim_{n\to\infty} x_n$. Finally show that F is not multiplicative in general, i.e. there are $x, y \in l^\infty(\mathbb{N})$ with $F(x \cdot y) \neq F(x) \cdot F(y)$.
Hint: Consider $p(x) := \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^n x_i$.