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Functional Analysis I

9th problem sheet

Please return your answers in the tutorials on June, 19th / 20th.

Problem 1: **4 pt.**

Let $(X, \|\cdot\|)$ be a reflexive Banach space and let (x_n) be a sequence in X and (f_n) be a sequence in the dual space X'. Show the following:

(a) If $x_n \rightharpoonup x$ in X and $f_n \to f$ in X^* then we have $f_n(x_n) \to f(x)$.

(b) If $x_n \to x$ in X and $f_n \overset{*}{\rightharpoonup} f$ in X^* then we have $f_n(x_n) \to f(x)$. Here $f_n \overset{*}{\rightharpoonup} f$ in X^* just means $f_n(x) \to f(x)$ for all $x \in X$.

(c) The proposition $f_n(x_n) \to f(x)$ does not hold if we only assume $x_n \rightharpoonup x$ in X and $f_n \overset{*}{\rightharpoonup} f$ in X^*.

Problem 2: **6 pt.**

Let $(X, \|\cdot\|)$ be a Banach space.

(i) Assume that X is reflexive. Show that all closed subspaces of X are reflexive.

(ii) Show that X is reflexive if and only if X' is reflexive.

(iii) Assume that X is reflexive. Show that every bounded sequence in X has a weakly convergent subsequence.

Remark: This was proven in the lecture (and you may use this) for the special case that X is separable.

Problem 3: **5 pt.**

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces.an let $A : X \times Y \to \mathbb{R}$ be bilinear. Assume

$$A(x, \cdot) \in Y' \text{ for all } x \in X \quad \text{and} \quad A(\cdot, y) \in X' \text{ for all } y \in Y.$$

(i) Show that there is a constant $c > 0$ such that

$$|A(x, y)| \leq c\|x\|\|y\|, \quad x \in X, y \in Y.$$

(This means A is a *bounded* bilinear form.)

(ii) Let $(x_n) \subset X$ and $(y_n) \subset Y$ be convergent sequences. Show that $A(x_n, y_n)$ converges in $\mathbb{R}$.

Problem 4: **5 pt.**

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be Banach spaces and let $T : X \to Y$ be a linear operator. T is called *strongly continuous* if (Tu_n) converges strongly to Tu in Y for all sequences (u_n) in X which converge weakly to $u \in X$, i.e. $u_n \rightharpoonup u$ in X implies $Tu_n \to Tu$ in Y.

Show:

(i) If T is compact then T is strongly continuous.

(ii) If T is strongly continuous and X is reflexive then T is compact.

Bonus problem 1: **5 Bonus pt.**

Let X be a nonempty topological space. Consider two players A and B playing the following infinite game:

A starts by choosing an arbitrary nonempty open set $V_0 \subset X$. Then B chooses a nonemtpy open set $V_1 \subset V_0$, whereupon A again chooses a nonempty open set $V_2 \subset V_1$ and so on. This results in a sequence of nested open sets $V_0 \supset V_1 \supset \cdots \supset V_n \supset \dots$.

The player B wins the game if $\bigcap V_n \neq \emptyset$. Otherwise A wins.

B is said to have a winning strategy if there is a mapping $\Phi : \tau \to \tau$ (where τ is the family of open nonempty sets) such that always $\Phi(V) \subset V$ and for every sequence $V_0 \supset V_1 = \Phi(V_0) \supset V_2 \supset V_3 = \Phi(V_2) \supset \dots$ we have $\bigcap V_n \neq \emptyset$.

Show the following:

(i) If X is a complete metric space then B has a winning strategy.

(ii) If B has a winning strategy then the Baire theorem holds in X.