Tutorials on May, 8th and 9th.

Problem 1:
Consider the Shift opertors on $l^2(\mathbb{N})$:

$$\begin{aligned} &S, T : l^2(\mathbb{N}) \to l^2(\mathbb{N}) \\ &x = (x_1, x_2, x_3, \dots) \mapsto Sx := (0, x_1, x_2, x_3, \dots) \\ &x = (x_1, x_2, x_3, \dots) \mapsto Tx := (x_2, x_3, x_4, \dots). \end{aligned}$$

Calculate the kernel, the range and the operator norm of both S and T.

Problem 2:
Let X be a vector space and let $|\cdot| : X \to [0, \infty)$ be a seminorm on X. We set

$$N := \{x \in X \,|\, |x| = 0\}.$$

(i) Show that N is a subspace of X.

Let $Y := X/N$ the quotient space induced by N. For $y \in Y$, $y = [x]$ define $\|y\| := |x|$.

(ii) Show that $\|\cdot\|$ is a norm on Y and $(Y, \|\cdot\|)$ is complete if X is.

Problem 3:
Consider $X = C^1([0, 1])$ equipped with the seminorm

$$|f| := \|f'\|_\infty, \quad f \in X$$

and

$$N := \{f \in X \,|\, |f| = 0\}.$$

Show that there is a bijective, linear mapping $T : Y \to U$ which maps the quotient space $Y = X/N$ onto the subspace

$$U := \{f \in X \,|\, f(0) = 0\}$$

such that

$$\|y\|_Y = |Ty|, \quad y \in Y.$$

That means that U and Y are isometrically isomorph.

Problem 4:
Let (X, d) be a normed space and let $Y \subset X$ be a linear subspace. For a given $x \in X$ the *best approximation* in Y is an element $\hat{y} \in Y$ such that

$$\|x - \hat{y}\| = d(x, Y) = \inf_{y \in Y} \|x - y\|.$$

(i) Show that best approximations need not be unique in general (but if $(X, \|\cdot\|)$ is strictly convex, uniqueness may be proven).

(ii) Show that best approximations need not exist: Consider $X = C([0, 1/2])$ and let Y be the subspace of all polynomials. Show there is no best approximation to $f \in X$, $f(t) = 1/(1 - t)$, $t \in [0, 1/2]$.