Technische Universität Berlin Sommer 2004
Fakultät II – Institut für Mathematik Stand 3. Mai 2004
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Sekretariat MA 6-2
`http://www.math.tu-berlin.de/Vorlesungen/SoSe04/KombGeoI/`

Third Problem Set 'Discrete Geometry'

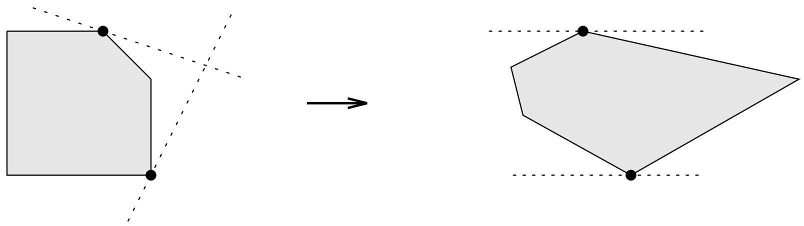
Projective Transformations, 3-Polytopes, Stellar Subdivisions

Deadline: *Wednesday, 12 May 2004*

Homework

1. Show that if $P \subseteq \mathbb{R}^d$ is a polytope with two distinct vertices u, v, then there is a projective transformation $P \leftrightarrow P'$ such that the vertices u' and v' have the smallest, respectively the largest, x_d-coordinate among all vertices of P'.

   Hint:



5 points

2. For every 3-polytope P, either the polytope P itself or its polar P^{Δ} has a facet that is a simplex.
   (This is not true for 4-polytopes — for these look at the regular 24-*cell*, whose facets are octahedra and whose vertex figures are cubes)

5 points

3. Let F be a facet of the d-polytope $P \subseteq \mathbb{R}^d$, and construct a point $y_F \in \mathbb{R}^d$ beyond F. The *stellar subdivision* of P at F is the polytope

$$\mathrm{st}\,(P, F) \quad := \quad \mathrm{conv}(P \cup \{y_F\}).$$

   (a) Describe the faces of $\mathrm{st}\,(P, F)$ in terms of faces of P. (This shows that the combinatorial type of $\mathrm{st}\,(P, F)$ does not depend on the precise position of y_F.)
   (b) Show that if P is simplicial, then so is $\mathrm{st}\,(P, F)$. In this case, count the number of k-faces of $\mathrm{st}\,(P, F)$ in terms of the numbers $f_i(P)$ of i-faces of P.
   (c) Describe the operation that is "polar" to stellar subdivision, given by

$$\mathrm{st}^{\Delta}(P, v) := (\mathrm{st}\,(P^{\Delta}, v^{\diamond}))^{\Delta},$$

   for any vertex v of P.

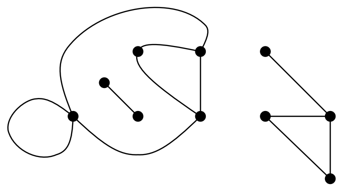
5 points

Σ 15 points

p.t.o.

Further Material

1. Let G be any finite graph drawn into the plane without crossings, and let v be its number of vertices, e its number of edges, c its number of connected components, and f the number of connected regions determined by it. For example, the graph

   has $v = 10$, $e = 11$, $f = 5$, and $c = 3$.

   Show that in general $v - e + f = 1 + c$. Deduce *Euler's formula*

   $$v - e + f = 2$$

   for the number of vertices v, the number of edges e, and the number of facets f of a 3-polytope.

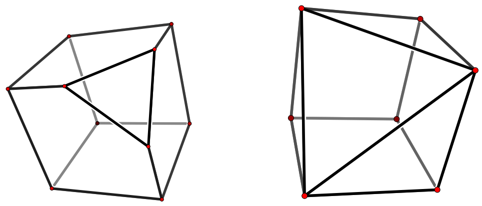
2. For which pairs (f_0, f_2) is there a 3-polytope?

3. Let L be a lattice, which has all the properties found in the lecture to be properties of the face lattice of a polytope. (L is graded, bounded, atomic, coatomic and has the diamond-property. Every interval of L and L^{op} shall have the same properties. The lattice stays connected when we take away $\hat{0}$ and $\hat{1}$.)

   (a) Show that if L has length 3 then L is isomorphic to the face lattice of a polytope.

   (b) Give a counter-example of length greater then 3.

4. In this exercise we compare the different proofs for Steinitz's theorem for the following examples:

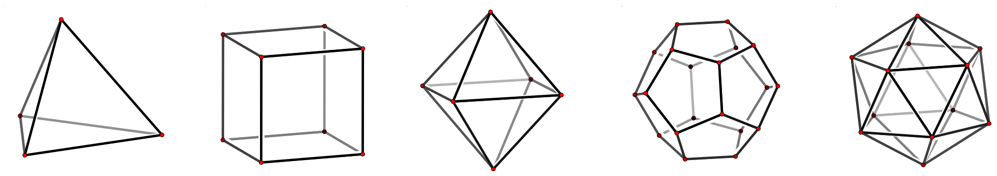
   "The cube with a vertex cut off" is a simple 3-polytope with 7 facets. If you cut off a vertex of the cube "completely" you get a 3-polytope with 7 vertices.

   (a) How many simple Delta-Wye transformations do you need to get to a simplex? If I admit only "cutting off a vertex" (that is, Wye-to-Delta transformations), how often do you have to polarize on the way?

   (b) Construct a (correct!) rubber band drawing.

   (c) Construct a (correct!?) planar circle packing representation.

5. A 3-Polytope is called *regular*, if every facet contains the same number of vertices and every vertex lies in the same number of facets. There are the following five regular polytopes, the platonic solids:



Simplex Cube Octahedron Dodecahedron Icosahedron

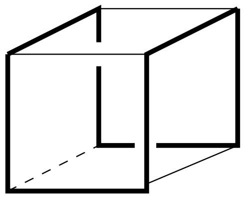
Show that there are no other regular 3-polytopes.

6. Classify up to combinatorial equivalence all polytopes that are the intersection of a 3-cube with a halfspace.

7. Show:
The cyclic 2-polytope $C_2(n) = \text{conv}\{c(0), \ldots, c(n-1)$ is affinely symmetric. That means: Find a bijective affine map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(c(i)) = f(c(n-1-i))$ for $i = 0, \ldots, n-1$.

What can you tell about cyclic polytopes of higher dimensions?

8. The graph of the 3-cube, $G(C_3)$, has cycles that go through all the vertices ("Hamiltonian cycles").



Take such a cycle, and then construct a realization of a combinatorial cube in 3-space such that the usual projection $\pi : \mathbb{R}^3 \leftrightarrow \mathbb{R}^2$ carries the cube to an 8-gon, and the cycle to the boundary cycle of the 8-gon. Then do the same for the dodecahedron.

9. Show that we cannot prescribe the *shape* of the shadow boundary of a 3-polytope.
Hint. You can follow Jürgen Richter-Gebert who noted that from a triangular prism, you can get a hexagon as a projection, but the shape of the hexagon cannot be prescribed.
Another way: Barnette gave a proof for the case where the 3-polytope is a tetrahedron with a stellar subdivision on every facet, and Q is a regular 8-gon.