TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

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http://www3.math.tu-berlin.de/Vorlesungen/SoSe13/MPIII/

Mathematical Physics III
SS 13

Exercise sheet 6

Homework (Deadline: **03.06**.2013, 14:15, at the beginning of the lecture)

H16. (60 points) Suppose that from some observations on a given physical system you end up with a mathematical model described by an ordinary differential equation. Such a differential equation describes the behavior of a continuous real-valued scalar function $u : X \to U \subset \mathbb{R}$. Here $x \in X \subset \mathbb{R}$ is your independent variable.

Unfortunately, your model is quite complicated. According to your observations, u obeys the following third-order nonlinear ordinary differential equation:

$$E := \partial_{111} u + 4\, x\, \partial_{11} u + \alpha\, \partial_1 u + \beta\, u = 0, \tag{1}$$

where $\alpha = \alpha(u)$ is an arbitrary function of u and $\beta = \beta(x)$ is an arbitrary function of x.

You are not able to solve this differential equation ... you try to solve it with some computer algebra system ... no way! It would be nice to find some special solutions, at least. OK ... what about Lie symmetries? That's a good idea. Lie symmetries can help you in understanding the structure of the equation itself. In particular, the existence of Lie symmetries might fix some constraints on your functions α and β. Furthermore - if you are lucky - symmetries might help you in reducing (1) to something of lower order, which is easier to solve.

Here is your homework: Construct the Lie symmetries (if any!), defined by

$$\mathbf{v} := \xi\, \partial_1 + \eta\, \not{\partial}, \qquad \xi, \eta \in \mathcal{F}(X \times U, \mathbb{R}),$$

of (1). Try to accomplish the following steps.

- *(20 pts)* Find the determining equations.
  (Hint: Some determining equations depend only on ξ and η (and their derivatives). Solve them, get a first ansatz for ξ and η and substitute this ansatz into the remaining determining equations)
- *(12 pts)* Find the most general admissible form of the function $\alpha = \alpha(u)$. This will depend on some real arbitrary coefficients.
- *(28 pts)* Solve the remaining set of determining equations, which now depend on the coefficients appearing in $\alpha = \alpha(u)$ and on $\beta = \beta(x)$. Collect all your conclusions.
  (Hint: At this stage some bifurcations occur! These bifurcations mainly depend on the values of the coefficients appearing in $\alpha = \alpha(u)$)

Homework policy
Homework assignments are due weekly at the beginning of the Tutorial (Monday, 14:15). They have to be turned in directly to the lecturer. No homework will be accepted after the deadline has passed.
Homework assignments can be to be solved in groups of two people.
To get the Übungsschein, you need to satisfactorily complete 60% of the homework assignments and to present the solution of one homework exercise at the blackboard in the tutorial.