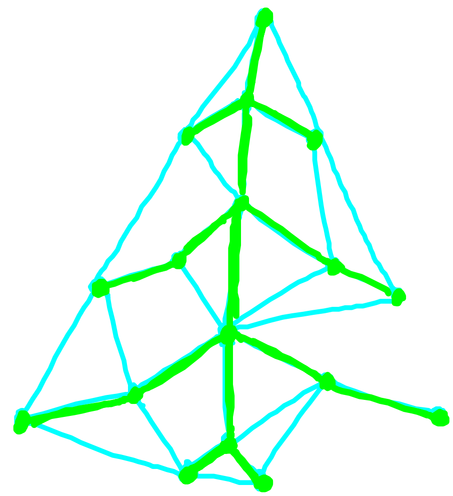
MST

Graph $G = (V, E)$, undirected, connected, costs $c_e \geq 0 \ \forall e \in E$



G

A spanning tree is a subgraph $T = (V, F)$, $F \subseteq E$, which is a tree (i.e. T is cycle-free & connected).

A minimal spanning tree (MST) is a spanning tree T with minimal cost $\sum_{e \in F} c_e$.

▸ LP:

$$\min \sum_{e \in E} c_e \cdot x_e$$

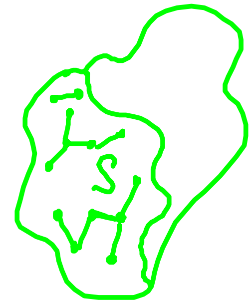
$$\text{s.t.} \sum_{e \in E} x_e = |V| - 1$$

$$\sum x_e \leq |S| - 1 \quad \forall \emptyset \neq S \subsetneq V$$

$$\sum_{e \in \gamma(S)}$$

$$x_e \geq 0$$

$$\gamma(S) := \{ e \in E \mid e = \{u,v\},\ u,v \in S \}$$



$\longrightarrow$ LP gives a lower bound on the cost of a MST.
i.e.: LP is a relaxation of MST.

- MST & TSP:

	MST	TSP
problem:	easy	NP-hard
LP relaxation:	exact	yields only lower bound
solve LP:	difficult (exponentially many constraints!)	by cutting plane method

- Let $T = (V, F)$ be a minimal spanning tree for $G = (V, E)$ and $x^* = (x^*_e)_{e \in E}$ with

$$x^*_e = \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{if } e \notin F \end{cases}.$$

Then x^* is an optimal solution of (LP).

<u>Proof.</u>

i. Reformulation of (LP)

(LP') $\min \sum_{e \in E} c_e x_e \iff \max \sum_{e \in E} -c_e x_e$

s.t. $\sum_{e \in E} x_e = |V| - 1$

$\sum_{e \in A} x_e \leq |V| - \varkappa(A) \quad \forall A \subsetneq E$

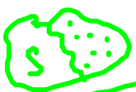
$x_e \geq 0$

(LP)

$\vdots$

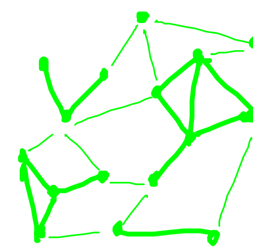
$\sum_{e \in \gamma(S)} x_e \leq |S| - 1$

$\forall S$



where $\varkappa(A)$ = # of connected components of the subgraph induced by A

(LP) is equivalent to (LP')!



(ii) Dualise (LP') $\longrightarrow$ (DP')

$\min \sum (|V| - \varkappa(A)) y_A$

s.t. $\sum_{A \ni e} y_A \geq -c_e \quad \forall e \in E$

$y_A \geq 0 \quad \forall A \subsetneq E$

y_E free

(iii) Suppose we have a solution y^* of (DP') that satisfies complementary slackness conditions together with x^*.

Then by compl. slack. both x^* and y^* are optimal for (LP'), (DP') resp.

(iv) Construct such a y^* by Kruskal's algorithm:

<u>Kruskal</u>

Input: Graph $G = (V, E)$

Output: MST $T = (V, F)$ for G

1. Order the edges $E = \{e_1, \dots, e_m\}$ such that $c_{e_1} \leq c_{e_2} \leq \dots \leq c_{e_m}$.
2. $T := (V, F)$ with $F := \emptyset$
3. For $i = 1$ to m

   If e_i connects different components of T

   Then $F = F \cup \{e_i\}$

$\rightarrow$ dual solution y^*:

$R_i := \{e_1, \dots, e_i\}$ (in the ordering of Kruskal's a.)

$1 \leq i \leq m$

$$\text{def} \quad y_A^* := \begin{cases} c_{e_{i+1}} - c_{e_i} & \text{if } A = R_i \ , 1 \le i < m \\ -c_{e_m} & \text{if } A = R_m = E \\ 0 & \text{if } A \ne R_i \ \forall i \end{cases}$$

- y^* is a solution of (DP') (order of e_i's !)

$$\sum_{e_k \in A} y_A^* = \sum_{i=k}^{m} y_{R_i}^* = (c_{e_{k+1}} - c_{e_k}) + (c_{e_{k+2}} - c_{e_{k+1}}) \dots - c_{e_m}$$

$$= - c_{e_k}$$

$y_A^* > 0$

$$\Rightarrow \sum_{e \in A} x_e^* = |V| - \kappa(A)$$

$\Rightarrow$ complementary slackness $\square$.