Exponentially many constraints

- The ellipsoid method shows that solving a LP can always be done in time polynomial in the input size.

  Even better: sometimes an LP can be solved in time polynomial in the number of variables, even if there are exponentially many constraints.

- Separation problem: Given a vector x

  either assert that x is feasible

  or find a violated constraint

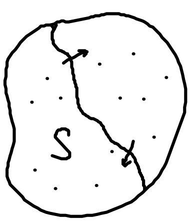
- TSP as linear program: $K_n = (V, E)$, costs $c_e \in \mathbb{R}$ $\forall e \in E$

$$\min \sum_{e \in E} c_e x_e$$

$$\text{s.t.} \sum_{e \in \delta(v)} x_e = 2 \qquad \forall v \in V \qquad \text{(n equalities)}$$

$$\sum_{e \in \delta(S)} x_e \geq 2 \qquad \forall S \subset V,\ S \neq \emptyset, V \quad (2^n - 2 \text{ inequalities})$$

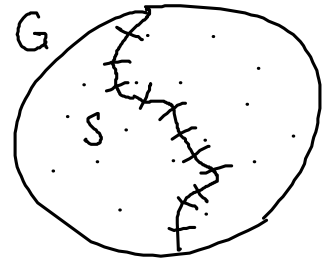
$$x_e \geq 0 \qquad \forall e \in E \qquad (n(n-1) \text{ variables})$$



Separation problem: Given $(x_e)_{e \in E} \geq 0$ with $\sum_{e \in \delta(v)} x_e = 2$ $\forall v \in V$

either find an $S \subset V$, $S \neq \emptyset, V$ with $\sum_{e \in \delta(S)} x_e < 2$

or prove that no such S exists



→ try to find the S with minimal $\sum_{e \in \delta(S)} x_e$

→ if this is ≥ 2 ⇒ all inequality constraints are satisfied

→ otherwise: we found such a set S !

→ Sepearation problem is a <u>min-cut problem</u> in G with x_e as edge capacities

and this is dual to the <u>max-flow problem</u>

▸ max flow & min cut: $G = (V, E)$ directed with edge capacities $u_e \ \forall e \in E$

choose $s, t \in V$, $s \neq t$

→ find a flow $(f_e)_{e \in E}$ that does not exceed the edge capacities and maximizes it's value

→ LP:

$$\max \sum_{e \in \delta^+(s)} f_e - \sum_{e \in \delta^-(s)} f_e$$

$$\text{s.t.} \quad \sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e = 0 \qquad \forall v \in V \setminus \{s, t\} \qquad \text{(flow conservation)}$$

$$f_e \leq u_e \qquad \forall e \in E$$

$$f_e \geq 0 \qquad \text{—"—}$$

$\Big\downarrow$ dualize

dual variables: p_v free $\quad \forall v \in V \setminus \{s,t\}$

$q_e \geq 0 \quad \forall e \in E$

$$\min \sum_{e \in E} u_e q_e$$

$$\text{s.t.} \quad p_u - p_v + q_e \geq 0 \qquad \forall e \in E,\ e = (u,v),\ u, v \notin \{s,t\}$$

$$-p_v + q_e \geq 1 \qquad \forall e = (s, v)$$

$$p_u + q_e \geq -1 \qquad \forall e = (u, s)$$

$$
\begin{aligned}
p_u \qquad + q_e &\geq 0 \qquad && \forall e = (u,t) \\
-p_v + q_e &\geq 0 && \forall e \in (t,v)
\end{aligned}
$$

introduce $p_s = -1$, $p_t = 0$

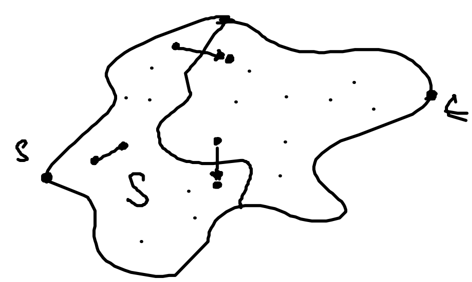
rewrite

$$
\begin{aligned}
\min \quad & \sum_{e\in E} u_e q_e \\
\text{s.t.} \quad & p_u - p_v + q_e \geq 0 && \forall e = [u,v] \in E \\
& p_s = 0 \\
& p_t = 1 \\
& p_v \text{ free} && \forall v \in V \setminus \{s,t\} \\
& q_e \geq 0 && \forall e \in E
\end{aligned}
$$

- this really calculates a minimal cut ! :

general observation: Every feasible flow has value at <u>most</u> the capacity of any s,t - cut.

Let S be an s,t - cut and $(f_e)_{e\in E}$ a feasible flow



$$
\begin{aligned}
\Rightarrow \sum_{e\in\delta^+(s)} f_e - \sum_{e\in\delta^-(s)} f_e
&= \sum_{v\in S}\Big(\sum_{e\in\delta^+(v)} f_e - \sum_{e\in\delta^-(v)} f_e\Big) \\
&= \underbrace{\sum_{e\in\delta^+(S)} f_e}_{\leq u_e} - \underbrace{\sum_{e\in\delta^-(S)} f_e}_{\geq 0} \leq \sum_{e\in\delta^+(S)} u_e
\end{aligned}
$$

□.

Let f^*, p^*, q^* be an optimal primal / dual solution.

$S := \{ v\in V \mid p_v^* < 1 \} \qquad \Rightarrow s \in S,\ t \notin S$

$$\Rightarrow \sum_{e\in\delta^+(s)} f_e^* - \sum_{e\in\delta^-(s)} f_e^* = \sum_{e\in\delta^+(S)} f_e^* - \sum_{e\in\delta^-(S)} f_e^* = \sum_{e\in\delta^+(S)} u_e \Rightarrow S \text{ is minimal}$$

$= u_e$ $\quad = 0$

$$e = (u,v) \in \delta^+(S) \quad \Rightarrow \quad q_e^* \geq \underset{\overset{\uparrow}{\geq 1}}{p_v^*} - \underset{\overset{\uparrow}{<1}}{p_u^*} > 0 \quad \xRightarrow{\text{comp. sl.}} \quad f_e^* = u_e$$

$$e = (u,v) \in \delta^-(S) \quad \Rightarrow \quad p_u^* - p_v^* + q_e^* \geq \underset{\overset{\uparrow}{\geq 1}}{p_u^*} - \underset{\overset{\uparrow}{<1}}{p_v^*} > 0 \quad \xRightarrow{\text{comp. sl.}} \quad f_e^* = 0$$