Exponentially many constraints

- The ellipsoid method shows that solving a LP can always be done in time polynomial in the input size.

  Even better: <u>sometimes</u> an LP can be solved in time polynomial in the number of variables, even if there are exponentially many constraints.

- <u>Separation problem</u>: Given a vector x

  <u>either</u> assert that x is feasible

  <u>or</u> find a violated constraint

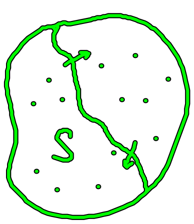
- TSP as linear program: $K_n = (V, E)$, costs $c_e \in \mathbb{R}$ $\forall e \in E$

$$\min \sum_{e \in E} c_e x_e$$

$$\text{s.t.} \quad \sum_{e \in \delta(v)} x_e = 2 \qquad \forall v \in V \qquad (n \text{ equalities})$$

$$\sum_{e \in \delta(S)} x_e \geq 2 \qquad \forall S \subset V,\ S \neq \emptyset, V \quad (2^n - 2 \text{ inequalities})$$

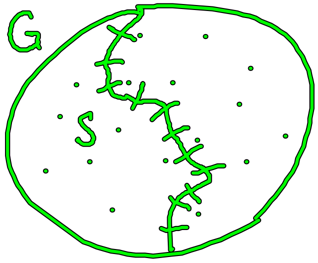
$$x_e \geq 0 \qquad \forall e \in E \qquad (n(n-1) \text{ variables})$$



Separation problem: Given $(x_e)_{e \in E} \geq 0$ with $\sum_{e \in \delta(v)} x_e = 2$ $\forall v \in V$

<u>either</u> find an $S \subset V$, $S \neq \emptyset, V$ with $\sum_{e \in \delta(S)} x_e < 2$

<u>or</u> prove that no such S exists



→ try to find the S with

minimal $\sum_{e \in \delta(S)} x_e$

→ if this is ≥ 2 ⇒ all inequality constraints are satisfied

→ otherwise: we found such a set S !

→ Seperation problem is a min-cut problem in G with x_e as edge capacities

and this is dual to the max-flow problem

▶ max flow & min cut: $G = (V, E)$ directed with edge capacities $u_e \quad \forall e \in E$

choose $s, t \in V$, $s \neq t$

→ find a flow $(f_e)_{e \in E}$ that does not exceed the edge capacities and maximizes it's value

→ LP:

$$\max \sum_{e \in \delta^+(s)} f_e - \sum_{e \in \delta^-(s)} f_e$$

$$\text{s.t.} \quad \sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e = 0 \qquad \forall v \in V \setminus \{s,t\} \qquad \text{(flow conservation)}$$

$$f_e \leq u_e \qquad \forall e \in E$$

$$f_e \geq 0 \qquad \text{—"—}$$

dualize

dual variables: p_v free $\quad \forall v \in V \setminus \{s,t\}$

$q_e \geq 0 \quad \forall e \in E$

$$\min \sum_{e \in E} u_e q_e$$

$$\text{s.t.} \quad p_u - p_v + q_e \geq 0 \qquad \forall e \in E,\ e = (u,v),\ u,v \notin \{s,t\}$$

$$-p_v + q_e \geq 1 \qquad \forall e = (s,v)$$

$$p_u + q_e \geq -1 \qquad \forall e = (u,s)$$

$$p_u \qquad + q_e \geq 0 \qquad \forall e = (u,t)$$

$$- p_v + q_e \geq 0 \qquad \forall e \in (t,v)$$

introduce $p_s = -1$, $p_t = 0$

rewrite

$$\min \sum_{e \in E} u_e q_e$$

$$\text{s.t.} \quad p_u - p_v + q_e \geq 0 \qquad \forall e = [u,v] \in E$$

$$p_s = 0$$

$$p_t = 1$$

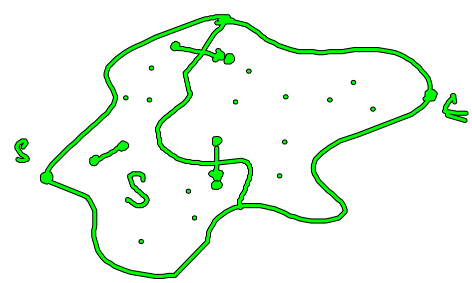
$$p_v \text{ free} \qquad \forall v \in V \setminus \{s,t\}$$

$$q_e \geq 0 \qquad \forall e \in E$$

▷ this really calculates a minimal cut !:

general observation: Every feasible flow has value at <u>most</u> the capacity of any s,t-cut.

Let S be an s,t-cut and $(f_e)_{e \in E}$ a feasible flow



$$\Rightarrow \sum_{e \in \delta^+(s)} f_e - \sum_{e \in \delta^-(s)} f_e$$

$$= \sum_{v \in S} \left(\sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e \right)$$

$$= \sum_{e \in \delta^+(S)} \underbrace{f_e}_{\leq u_e} - \sum_{e \in \delta^-(S)} \underbrace{f_e}_{\geq 0} \leq \sum_{e \in \delta^+(S)} u_e$$

□

Let f^*, p^*, q^* be an optimal primal / dual solution.

$$S := \{ v \in V \mid p_v^* < 1 \} \quad \Rightarrow s \in S, \ t \notin S$$

$$\Rightarrow \sum_{e \in \delta^+(s)} f_e^* - \sum_{e \in \delta^-(s)} f_e^* = \sum_{e \in \delta^+(S)} f_e^* - \sum_{e \in \delta^-(S)} f_e^* = \sum_{e \in \delta^+(S)} u_e \Rightarrow S \text{ is minimal}$$

$= u_e \qquad = 0$

$$e = (u,v) \in \delta^+(S) \quad \Rightarrow \quad q_e^* \geq \underset{\geq 1}{\underset{\uparrow}{p_v^*}} - \underset{< 1}{\underset{\uparrow}{p_u^*}} > 0 \quad \xRightarrow{\text{comp. sl.}} \quad f_e^* = u_e$$

$$e = (u,v) \in \delta^-(S) \quad \Rightarrow \quad p_u^* - p_v^* + q_e^* \geq \underset{\geq 1}{\underset{\uparrow}{p_u^*}} - \underset{< 1}{\underset{\uparrow}{p_v^*}} > 0 \quad \xRightarrow{\text{comp. sl.}} \quad f_e^* = 0$$