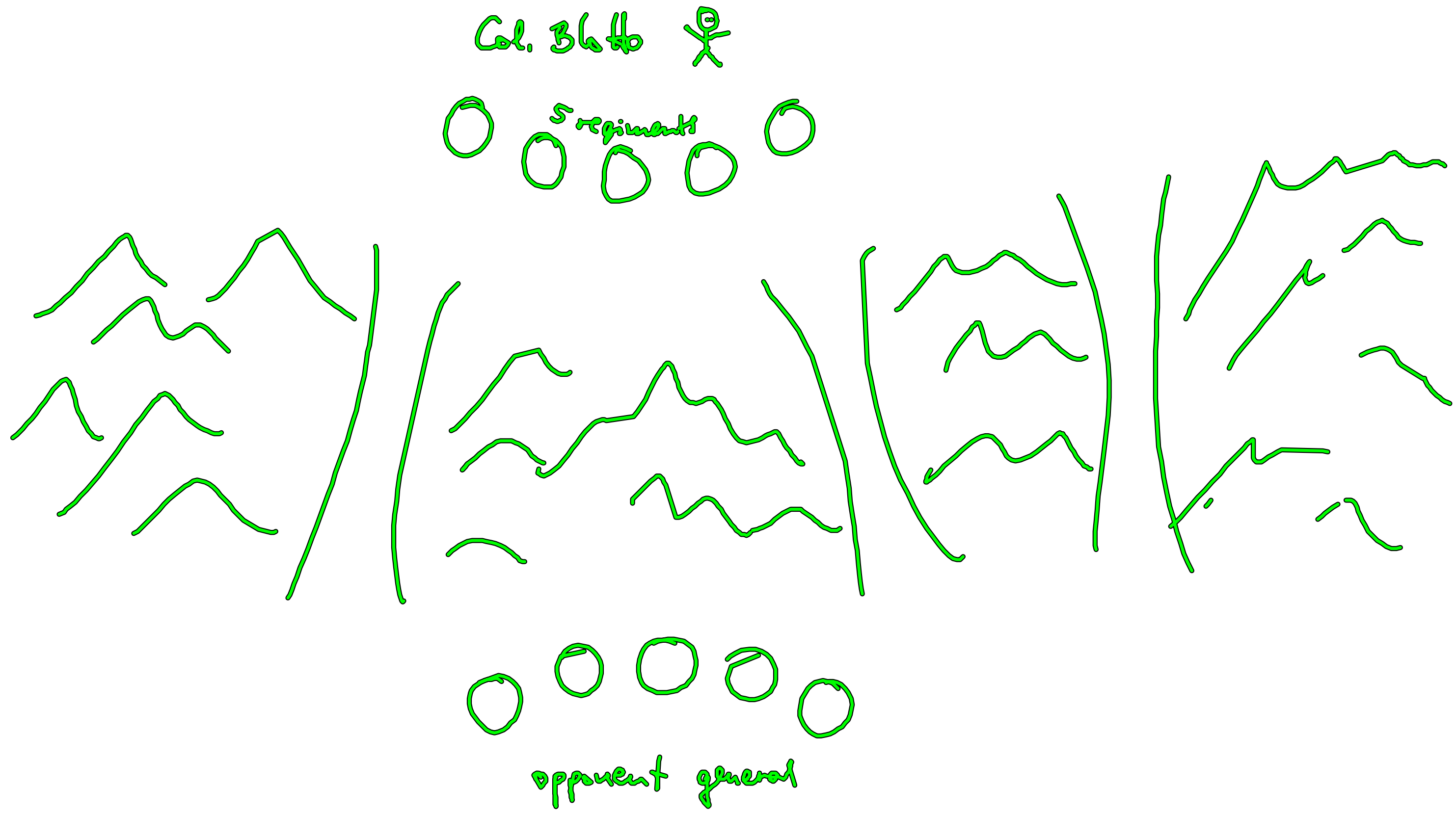
Zero-sum matrix games

- „Game" (Borel, 1921)



→ a mountain pass is won by the commander who sends more regiments to it than the other;
the battle is won by the one who has conquered more passes than the other

→ partition the number of regiments:
$(5,0,0)$, $(4,1,0)$, $(3,2,0)$, $(3,1,1)$, $(2,2,1)$
assign them in one of $3! = 6$ possible ways to the passes randomly

→ winning probabilities:
e.g. with $(4,1,0)$ for Col. Blotto
and $(5,0,0)$ for the opponent,
the winning probability for Col. Blotto is $\frac{1}{3}$ and with
prob. $\frac{2}{3}$ there will be a draw $\Rightarrow$ payoff: $\frac{1}{3} - 0 = \frac{1}{3}$

→ payoff matrix (for Col. Blotto)

Strategies of Col. Blotto

	(5,0,0)	(4,1,0)	(3,2,0)	(3,1,1)	(2,2,1)
→ (5,0,0)	0	$-1/3$	$-1/3$	-1	-1
→ (4,1,0)	$1/3$	0	0	$-1/3$	$-2/3$
→ (3,2,0)	$1/3$	0	0	0	$1/3$
→ (3,1,1)	1	$1/3$	0	0	$-1/3$
→ (2,2,1)	1	$2/3$	$-1/3$	$1/3$	0

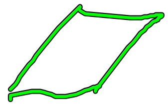
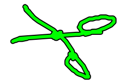
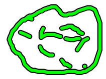
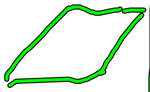
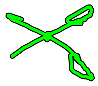
→ choose a strategy that guarantees the <u>highest payoff</u> in the worst case (≙ smallest value in the row): $(3,2,0)$

→ opponent: chooses a strategy that guarantees the <u>lowest payoff</u> in his worst case (≙ highest value in the column): $(3,2,0)$

→ strategies are <u>best responses</u> to each other (<u>Nash equilibrium</u>)

▸ another example: Rock, paper, scissors

→ payoff matrix: player B

player A				
		0	-1	1
		1	0	-1
		-1	1	0

→ no (pure) Nash equilibrium!

→ solution: choose every strategy with same probability: $\frac{1}{3}$

→ no reason to change the $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$-strategy

→ mixed Nash equilibrium

mixed strategy

- In general: two players A and B with n and m pure strategies, resp.

The payoff matrix is $M := (m_{i,j})_{\substack{1 \le i \le n \\ 1 \le j \le m}}$ where

m_{ij} gives the payoff for A if strategy i is played against str. j.

A mixed strategy is a probability distribution over the pure strategies:

for A: $x = (x_1, \dots, x_n) \geq 0$, $\sum_{i=1}^{n} x_i = 1$

for B: $y = (y_1, \dots, y_m) \geq 0$, $\sum_{j=1}^{m} y_j = 1$

Given two mixed strat. x and y, the expected payoff for A is:

$$\sum_{i,j} m_{ij} \cdot P_{x,y} [A \text{ plays } i, B \text{ plays } j]$$

$$= \sum_{i,j} m_{ij} \cdot P_x [A \text{ plays } i] \cdot P_y [B \text{ plays } j]$$

$$= \sum_{i,j} m_{ij} \cdot x_i \cdot y_j$$

$$= x^T M y$$

→ If Player A plays strategy x, the worst case payoff is

$$\min_y x^T M y$$

⇒ Player A plays the strategy $\tilde{x}$ that maximises $\min_y x^T M y$

→ Player B plays the strategy $\tilde{y}$ that minimises $\max_x x^T M y$

Then: $$\min_y \tilde{x}^T M y \leq \tilde{x}^T M \tilde{y} \leq \max_x x^T M \tilde{y}$$

<u>Theorem.</u> (Minimax Theorem (J. v. Neumann, 1926) for zero-sum matrix games)

For every $n \times m$ payoff matrix M there exist mixed strategies $\tilde{x}$ and $\tilde{y}$ such that

$$\max_x \min_y x^T M y = \min_y \tilde{x}^T M y = \tilde{x}^T M \tilde{y} = \max_x x^T M \tilde{y} = \min_y \max_x x^T M y .$$

$(\tilde{x}, \tilde{y})$ is called a <u>Nash equilibrium</u>, $\tilde{x}^T M \tilde{y}$ is the <u>game value</u>.

<u>Proof.</u> Define $\tilde{x}$ and $\tilde{y}$ as above.

B's best response to a given mixed strategy x of A is the solution of the linear program

$$\begin{aligned} \min \quad & x^T M y \\ \text{s.t.} \quad & \sum y = 1 \\ & y \geq 0 \end{aligned}$$

Let $\beta(x)$ be the solution of this linear program.

A's task is to maximise $\beta(x)$ over all his strategies x

Dualize the above LP:

$$\begin{aligned} \max \quad & x_0 \\ \text{s.t.} \quad & M^T x - \mathbb{1} x_0 \geq 0 \end{aligned}$$

This also has optimal solution $\beta(x)$!

Extend this:

$$(*) \quad \begin{aligned} \max \quad & x_0 \\ \text{s.t.} \quad & M^T x - \mathbb{1} x_0 \geq 0 \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_1, \dots, x_n \geq 0 \end{aligned}$$

The optimal solution to $(*)$ is $\tilde{x}$ with optimal value $\tilde{x}_0$.

Do the same for player B ... then $\tilde{y}$ is the optimal solution to the lin. program

$$(**) \quad \begin{aligned} \min \quad & y_0 \\ \text{s.t.} \quad & M y - \mathbb{1} y_0 \leq 0 \\ & \sum_{j=1}^{m} y_j = 1 \\ & y_1, \dots, y_m \geq 0 \end{aligned}$$

$(*)$ and $(**)$ are dual to each other

$\Rightarrow \tilde{x}_0 = \tilde{y}_0$ (by the duality theorem) $\square$.