

# Chapter 1: Introduction

## 1.1 Variants of the linear programming problem

### Example:

$$\text{minimize } 2x_1 - x_2 + 4x_3$$

$$\text{subject to: } x_1 + x_2 + x_4 \leq 2$$

$$3x_2 - x_3 = 5$$

$$x_3 + x_4 \geq 3$$

$$x_1 \geq 0$$

$$x_3 \leq 0$$

linear cost function  
" objective "

linear equality  
and inequality  
constraints

special constraints

### General linear program

Given:  $c \in \mathbb{R}^n$ ,  $a_i \in \mathbb{R}^n$   $i \in M_1 \cup M_2 \cup M_3$ ,  $b_i \in \mathbb{R}$   $i \in M_1 \cup M_2 \cup M_3$

Task: Find  $x \in \mathbb{R}^n$

finite index sets

$$\text{min } c^T x$$

$$\text{s.t. } \left. \begin{array}{l} a_i^T \cdot x \geq b_i \quad \text{for } i \in M_1 \\ a_i^T \cdot x = b_i \quad \text{for } i \in M_2 \\ a_i^T \cdot x \leq b_i \quad \text{for } i \in M_3 \\ x_j \geq 0 \quad \text{for } j \in N_1 \\ x_j \leq 0 \quad \text{for } j \in N_2 \end{array} \right\} (*)$$

$x$  is called vector  
of (decision) variables.

$$N_1, N_2 \subseteq \{1, \dots, n\}$$

$x$  satisfying (\*) is feasible solution

Feasible solution  $x^*$  with  $c^T x^* \leq c^T x \quad \forall$  feasible  $x$   
is an optimal solution.

The problem is unbounded if  $\forall k \in \mathbb{R} \exists$  feasible  $x$ :  
 $c^T x \leq k$

Remark:  $\max c^T x \iff \min (-c)^T \cdot x$

Remark: Any linear program (LP) can be written in  
the following form:

$$\text{min } c^T x$$

$$\text{s.t. } A \cdot x \geq b$$

for some  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$

$$A = \begin{pmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_m \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

(Rewrite  $a_i^T \cdot x = \delta_i$  as  $a_i^T x \geq \delta_i \wedge -a_i^T \cdot x \geq -\delta_i$   
 $a_i^T \cdot x \leq \delta_i$  as  $-a_i^T x \geq -\delta_i$ )

## Standard form problems

LP in standard form

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned} \quad c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

Example: The "diet problem"

Given:  $n$  different foods,  $m$  different nutrients  
 $a_{ij}$  = amount of nutrient  $i$  in one unit of food  $j$   
 Let  $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$  where  $b_i$  is requirement of nutrient  $i$  in some ideal diet.

Task: Find an ideal diet consisting of foods  $1, \dots, n$

$$A \cdot x = b \quad \text{where } x_j = \# \text{ units of food } j \\ x \geq 0$$

$$\begin{aligned} A \cdot x &\geq b \\ x &\geq 0 \end{aligned} \quad \min c^T x$$

Reduction to standard form:

i) Elimination of free variables:

Replace  $x_j$  with  $x_j^+, x_j^- \geq 0$ :  $x_j := x_j^+ - x_j^-$

ii) Elimination of inequality constraints:

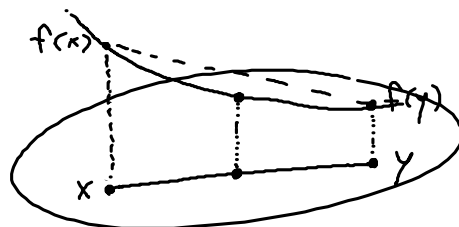
$$a^T x \leq b \rightarrow a^T x + s = b \quad s \geq 0 \text{ "slack variable"}$$

$$\bar{a}^T x \geq \bar{b} \rightarrow \bar{a}^T x - \bar{s} = \bar{b} \quad \bar{s} \geq 0$$

## 1.3 Piecewise linear convex objective functions

Def: i)  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is called convex if

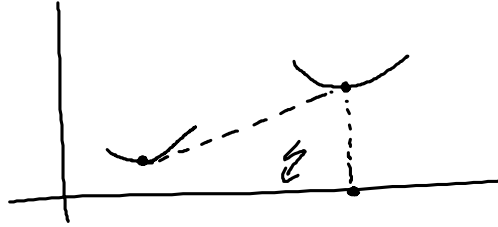
$$f(\lambda \cdot x + (1-\lambda) \cdot y) \leq \lambda \cdot f(x) + (1-\lambda) \cdot f(y) \quad \forall x, y \in \mathbb{R}^n, \lambda \in [0, 1]$$



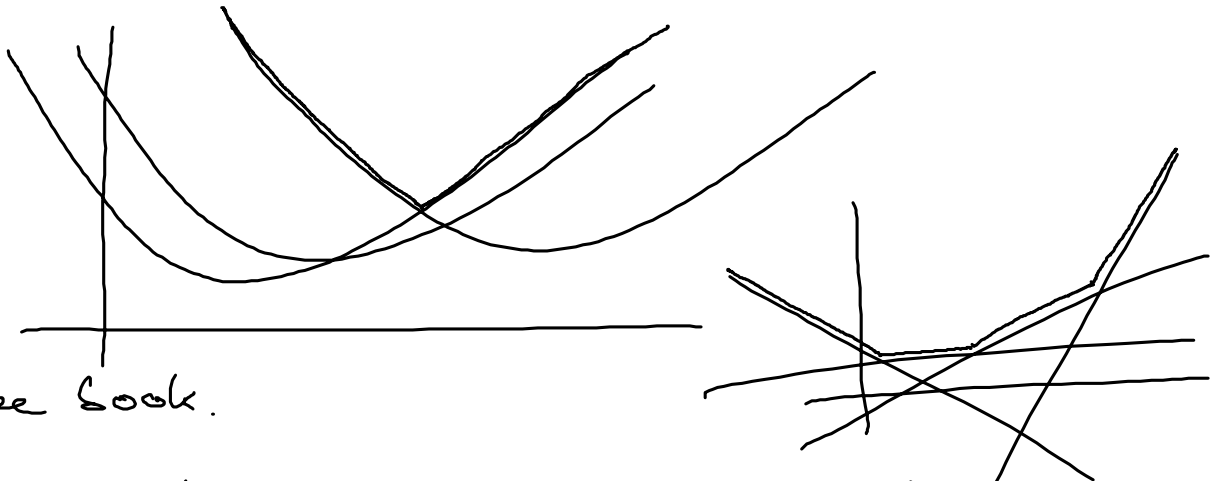
Notice:  $f$  convex  $\Leftrightarrow -f$  concave.

Example:  $f(x) = a_0 + \sum_{i=1}^n a_i \cdot x_i$  (affine linear) is both convex and concave

Remark: Convex functions play an important role in optimization because every local minimum is a global minimum.



Theorem: If  $f_1, \dots, f_k : \mathbb{R}^n \rightarrow \mathbb{R}$  are convex, then  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by  $f(x) := \max_{i=1, \dots, k} f_i(x)$  is also convex



Proof: see book.

Important special case:  $x \mapsto \max_{i=1, \dots, m} (c_i^T x + d_i)$

$$\min \max_{i=1, \dots, m} (c_i^T x + d_i)$$

$$\text{s.t. } Ax \geq \delta$$

$$\Leftrightarrow \min z$$

$$\text{s.t. } z \geq c_i^T x + d_i \quad \forall i=1, \dots, m$$

$$Ax \geq \delta$$

Example:  $c_1, \dots, c_n \geq 0$

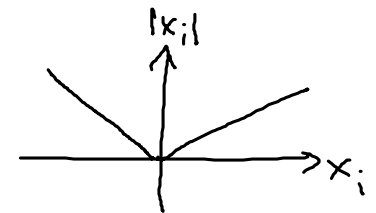
$$\min \sum_{i=1}^n c_i \cdot |x_i| \quad \Leftrightarrow \min \sum_{i=1}^n c_i \cdot z_i$$

$$\text{s.t. } Ax \geq \delta$$

$$\text{s.t. } x_i \leq z_i$$

$$-x_i \leq z_i$$

$$Ax \geq \delta$$



$$\Leftrightarrow \min \sum_{i=1}^n c_i \cdot (x_i^+ + x_i^-)$$

$$\text{s.t. } A \cdot x^+ - A \cdot x^- \geq \delta$$

## 1.4 Graphical representation and solution

Example:  $c = (-1)$

$$\min -x_1 - x_2$$

$$\text{s.t. } \boxed{x_1 + 2x_2 \leq 3}$$

$$\boxed{2x_1 + x_2 \leq 3}$$

$$\boxed{x_1, x_2 \geq 0}$$

