Chapter 1: Introduction

1.1 Variants of the linear programming problem

Example:

$$
\begin{array}{lrl}
\text{minimize} & 2x_1 - x_2 + 4x_3 & \\
\text{subject to:} & x_1 + x_2 \qquad + x_4 & \leq 2 \\
& 3x_2 - x_3 \qquad & = 5 \\
& x_3 + x_4 & \geq 3 \\
& x_1 & \geq 0 \\
& x_3 & \leq 0
\end{array}
$$

linear cost function
" objective "

linear equality and inequality constraints

special constraints

General linear program

Given: $c \in \mathbb{R}^n$, $a_i \in \mathbb{R}^n$ $i \in M_1 \dot\cup M_2 \dot\cup M_3$, $b_i \in \mathbb{R}$ $i \in M_1 \cup M_2 \cup M_3$

finite index sets

Task: Find $x \in \mathbb{R}^n$

$$
\begin{array}{llll}
\min & c^T x & & \\
\text{s.t.} & a_i^T \cdot x \geq b_i & \text{for } i \in M_1 & \\
& a_i^T \cdot x = b_i & \text{for } i \in M_2 & \\
& a_i^T \cdot x \leq b_i & \text{for } i \in M_3 & \\
& x_j \geq 0 & \text{for } j \in N_1 & \\
& x_j \leq 0 & \text{for } j \in N_2 &
\end{array}
\Bigg\} (*)
$$

x is called vector of (decision) variables.

$N_1, N_2 \subseteq \{1, \dots, n\}$

x satisfying $(*)$ is <u>feasible solution</u>

Feasible solution x^* with $c^T x^* \leq c^T x$ $\forall$ feasible x is an <u>optimal solution</u>.

The problem is unbounded if $\forall k \in \mathbb{R}$ $\exists$ feasible x: $c^T x \leq k$

Remark: $\max c^T x \iff \min (-c)^T \cdot x$

Remark: Any linear program (LP) can be written in the following form:

$$
\begin{array}{ll}
\min & c^T x \\
\text{s.t.} & A \cdot x \geq b
\end{array}
\qquad
A = \begin{pmatrix} - a_1 - \\ - a_2 - \\ \vdots \\ - a_m - \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}
$$

for some $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

(Rewrite $a_i^T \cdot x = b_i$ as $a_i^T x \ge b_i \wedge -a_i^T \cdot x \ge -b_i$
$a_i^T \cdot x \le b_i$ as $-a_i^T x \ge -b_i$)

Standard form problems

LP in standard form

$$\min c^T x \qquad c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$
$$\text{s.t. } A \cdot x = b$$
$$x \ge 0$$

Example: The "diet problem"

Given: n different foods, m different nutrients

a_{ij} = amount of nutrient i in one unit of food j

Let $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ where b_i is requirement of nutrient i in some ideal diet.

Task: Find an ideal diet consisting of foods $1, \dots, n$

$$A \cdot x = b \quad \text{where } x_j = \# \text{units of food } j$$
$$x \ge 0$$

$$A \cdot x \ge b \qquad \min c^T x$$
$$x \ge 0$$

Reduction to standard form:

i) Elimination of free variables:
Replace x_j with $x_j^+, x_j^- \ge 0$: $x_j := x_j^+ - x_j^-$

ii) Elimination of inequality constraints:

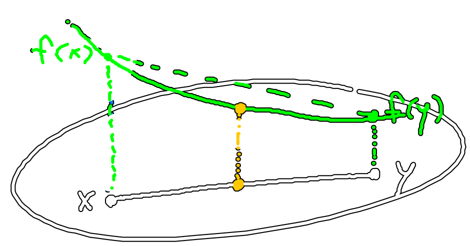
$$a^T x \le b \quad \longrightarrow \quad a^T x + s = b$$
$$s \ge 0 \quad \text{"slack variable"}$$

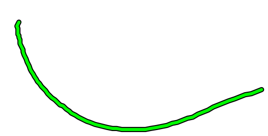
$$\bar{a}^T x \ge \bar{b} \quad \longrightarrow \quad \bar{a}^T x - \bar{s} = \bar{b}$$
$$\bar{s} \ge 0$$

1.3 Piecewise linear convex objective functions

Def: i) $f: \mathbb{R}^n \to \mathbb{R}$ is called convex (concave) if

$$f(\lambda \cdot x + (1-\lambda) \cdot y) \underset{(\ge)}{\le} \lambda \cdot f(x) + (1-\lambda) \cdot f(y) \quad \forall x, y \in \mathbb{R}^n, \lambda \in [0,1]$$

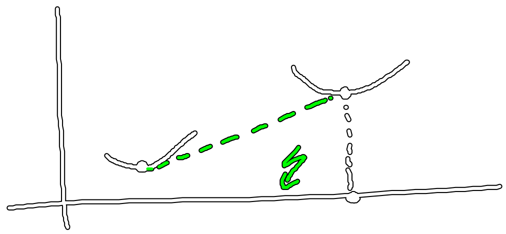




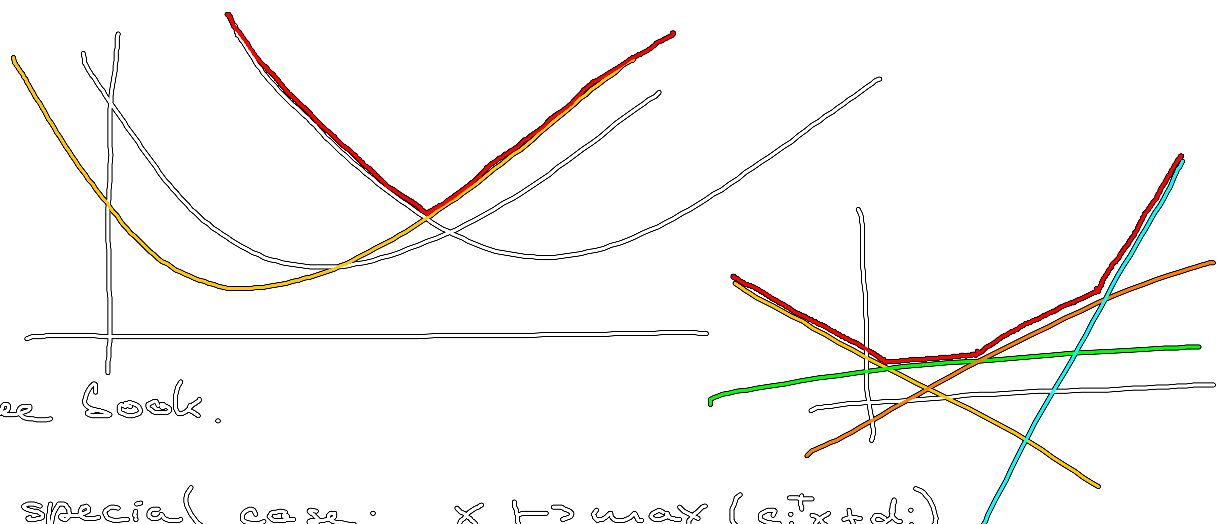
Notice: f convex $\Leftrightarrow$ $-f$ concave.

Example: $f(x) = a_0 + \sum_{i=1}^{n} a_i \cdot x_i$ (affine linear) is both convex and concave

Remark: Convex functions play an important role in optimization because every local minimum is a global minimum.



Theorem: If $f_1, \dots, f_k : \mathbb{R}^n \to \mathbb{R}$ are convex, then $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) := \max_{i=1,\dots,k} f_i(x)$ is also convex



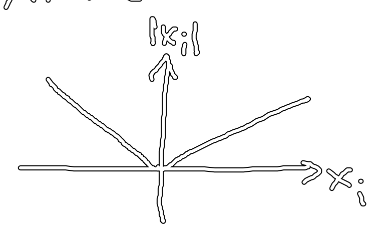
Proof: see book.

Important special case: $x \mapsto \max_{i=1,\dots,m} (c_i^T x + d_i)$

$$\begin{aligned} \min\ & \max_{i=1,\dots,m} (c_i^T x + d_i) \\ \text{s.t.}\ & Ax \geq b \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min\ & z \\ \text{s.t.}\ & z \geq c_i^T x + d_i \quad \forall i = 1,\dots,m \\ & Ax \geq b \end{aligned}$$

Example: $c_1, \dots, c_n \geq 0$

$$\begin{aligned} \min\ & \sum_{i=1}^{n} c_i \cdot |x_i| \\ \text{s.t.}\ & Ax \geq b \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min\ & \sum_{i=1}^{n} c_i \cdot z_i \\ \text{s.t.}\ & x_i \leq z_i \\ & -x_i \leq z_i \\ & A \cdot x \geq b \end{aligned}$$



$$\Longleftrightarrow \min \sum_{i=1}^{n} c_i \cdot (x_i^+ + x_i^-)$$

$$\text{s.t. } A \cdot x^+ - A \cdot x^- \geq b$$

1.4 Graphical representation and solution

Example:

$$c = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\min \ -x_1 - x_2$$

s.t. $\boxed{x_1 + 2x_2 \leq 3}$

$\boxed{2x_1 + x_2 \leq 3}$

$\boxed{x_1, x_2 \geq 0}$