2.3 Polyhedra in standard form

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ $\quad P = \{x \in \mathbb{R}^n \mid Ax = b, \ x \geq 0\}$

w.l.o.g: rows of A are linearly indep.
that is: $\operatorname{rank}(A) = m$

Theorem: $x \in \mathbb{R}^n$ is basic solution $\Leftrightarrow$

$Ax = b$ and there exist indices $B(1), \dots, B(m)$ $\in \{1, \dots, n\}$ such that

(i) columns $A_{B(1)}, \dots, A_{B(m)}$ are linearly indep.

(ii) $x_i = 0$ for $i \neq B(1), \dots, B(m)$.

Proof sketch:

$$Ax = b$$
$$I_n x \geq 0$$

n linearly indep. active constraints means:
m from $Ax = b$ plus $n-m$ from $I \cdot x \geq 0$:

$$Ax = b$$
$$x_j = 0 \quad \text{for } j \neq B(1), \dots, B(m)$$

These constraints are linearly independent
$\Leftrightarrow A_{B(1)}, \dots, A_{B(m)}$ linearly independent.

Assume w.l.o.g $B(i) = i$ for $i = 1, \dots, m$

$$\left(\begin{array}{ccc|ccc} A_{B(1)} & \cdots & A_{B(m)} & A_{m+1} & \cdots & A_n \\ \hline 0 & \cdots & 0 & & I_{n-m} & \end{array} \right)$$

$\square$

Let $B(1), \dots, B(m)$ as in the theorem and x

the corresp. basic solution. Then

$x_{B(1)}, \dots, x_{B(m)}$ are basic variables, the remain. variables are non-basic.

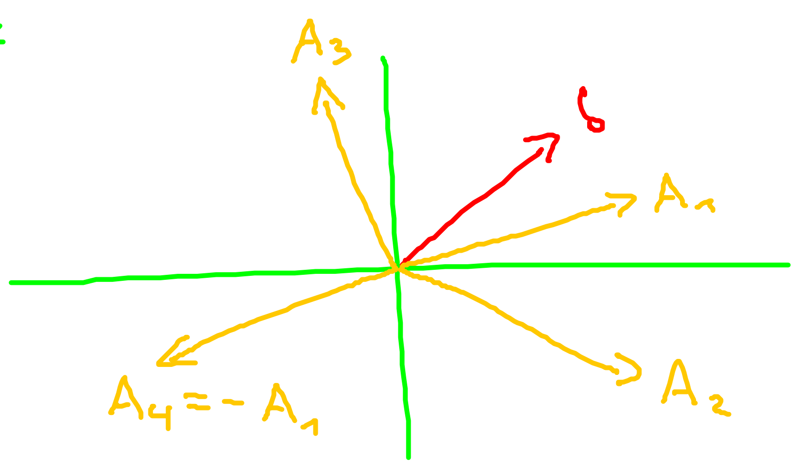
The columns $A_{B(1)}, \dots, A_{B(m)}$ are the basic columns, they form basis of $\mathbb{R}^m$.

Let $B = \left(A_{B(1)} \dots A_{B(m)} \right) \in \mathbb{R}^{m \times m}$ and

$x_B = \begin{pmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{pmatrix}$, then $B \cdot x_B = b$ and thus

$x_B = B^{-1} \cdot b$. In particular x is basic feasible solution if and only if $x_B = B^{-1} \cdot b \geq 0$

Example:



A_1, A_2 form basis but the corresp. basic sol. x is not feas. since $x_2 < 0$.

A_1, A_3 or A_2, A_3 form bases with corresp. basic feasible solutions.

A_1, A_4 do not form a basis.

Observation: Every basis $A_{B(1)}, \dots, A_{B(m)}$ determines a unique basic solution. Thus different basic sol. correspond to diff. bases.

But: Two different bases might yield the same basic solution. (e.g.: if $b=0$ then $x=0$ is the only basic solution.)

Def: Two bases $A_{B(1)}, \dots, A_{B(m)}$ and $A_{B'(1)} \dots A_{B'(m)}$ are *adjacent* if they share all but one column.

Observation: Adjacent basic solutions can always be obtained from two adjacent bases. If two adjacent bases lead to distinct basic solutions, then the latter are adjacent.

2.4. Degeneracy

Def: A basic solution $x \in \mathbb{R}^n$ is *degenerate* if more than n constraints are active at x.

Remark:

(i) A basic solution x of a polyhedron in standard form $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ is degenerate $\Leftrightarrow$ more than $n-m$ components of x are zero.

(ii) For a non-degenerate basic solution there is a unique basis.

There are <u>3 different reasons</u> for degeneracy

1) redundant variables:

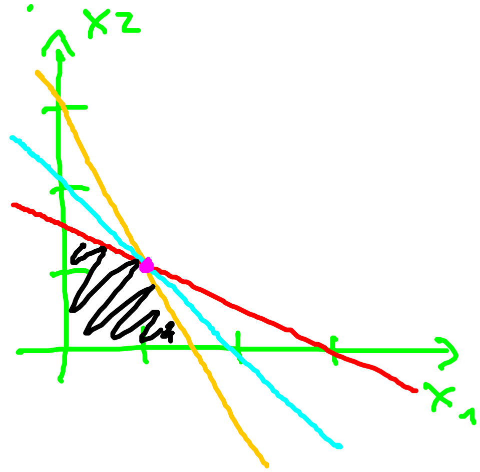
<u>Example</u>:

$$\begin{aligned} x_1 + x_2 \quad &= 1 \\ x_3 &= 0 \\ x_1, x_2, x_3 &\geq 0 \end{aligned} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2) redundant constraints:

<u>Example</u>:

$$\begin{aligned} x_1 + 2x_2 &\leq 3 \\ 2x_1 + x_2 &\leq 3 \\ x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

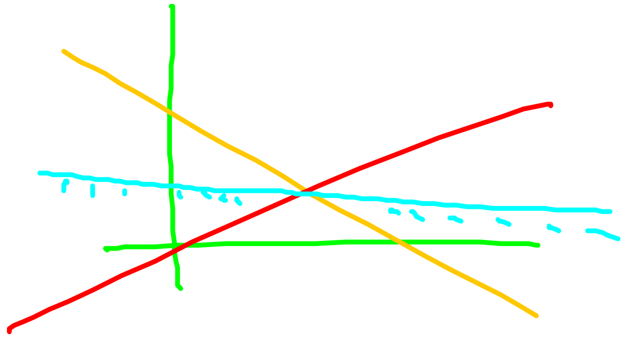


3) <u>Geometric reasons</u>:

<u>Example</u>: octahedron

<u>Observation</u>: Perturbing the right hand

side vector b may remove degeneracy.



2.5 Existence of extreme points

Def: A polyhedron $P \subseteq \mathbb{R}^n$ contains a line if there is $x \in P$ and $d \in \mathbb{R}^n \setminus \{0\}$ such that $x + \lambda \cdot d \in P \ \forall \lambda \in \mathbb{R}$

A polyhedron that does not contain an extreme point contains a line and vice versa