$P = \{x \in \mathbb{R}^n \mid A \cdot x = b, x \geq 0\}$ $\quad A \in \mathbb{R}^{m \times n}$, rank$(A) = m$

basic solution can be specified by choosing m linearly indep. columns of A: $\underbrace{(A_{B(1)}, \ldots, A_{B(m)})}_{} =: B$

$$x_B = \begin{pmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{pmatrix} = B^{-1} \cdot b \quad , \quad x_j = 0 \text{ for } j \neq B(1), \ldots, B(m)$$

$$A \cdot x = B \cdot x_B + \sum_{j \neq B(1), \ldots, B(m)} A_j \cdot x_j = B \cdot B^{-1} \cdot b + 0 = b$$

$$\| \\ \sum_{j=1}^{n} A_j \cdot x_j$$

2.5 Existence of extreme points

Def: A polyhedron P contains a line if there is a $x \in P$ and some direction $d \neq 0$ such that

$$x + \lambda \cdot d \in P \quad \forall \lambda \in \mathbb{R}.$$

Theorem: Let $P = \{x \in \mathbb{R}^n \mid Ax \geq b\} \neq \emptyset$. The following are equivalent:

(i) There is an extreme point $x \in P$

(ii) P does not contain a line

(iii) matrix A contains n linearly independent rows.

Proof: (i) $\Rightarrow$ (iii): Clear by def. of basic feasible solutions.

(iii) $\Rightarrow$ (ii): Assume by contradiction that $x \in P$ and

$$x + \lambda \cdot d \in P \quad \forall \lambda \in \mathbb{R} \qquad \text{for some } d$$

$$\Rightarrow A \cdot (x + \lambda \cdot d) = A \cdot x + \lambda \cdot A \cdot d \geq b \quad \forall \lambda \in \mathbb{R}$$

$$\Rightarrow A \cdot d = 0$$

Since rank$(A) = n$ $\Rightarrow d = 0$

(ii) $\Rightarrow$ (i): Let $x \in P$ such that $|\overbrace{\{i \mid a_i^T \cdot x = b_i\}}^{I :=}|$ is maximum.

Assume by contradiction that $\{a_i \mid a_i^T x = b_i\}$ does not contain n linearly indep. vectors.

$\Rightarrow \exists\, d \in \mathbb{R}^n \setminus \{0\}$ such that $a_i^T \cdot d = 0 \quad \forall i \in I$

$\Rightarrow a_i^T \cdot (x + \lambda \cdot d) = b_i \quad \forall \lambda \in \mathbb{R}$

Since $x+\lambda\cdot d$ is not contained in P for all $\lambda\in\mathbb{R}$, there is a constraint $a_j^T x \geq b_j$ that is eventually violated for some $\lambda\in\mathbb{R}$. $\Rightarrow \exists\, \lambda_0\in\mathbb{R}$ with

$$a_j^T(x+\lambda_0\cdot d) = b_j \quad \text{and} \quad x+\lambda_0\cdot d \in P.$$

This contradicts the choice of x. $\square$

<u>Corollary:</u> Every non-empty bounded polyhedron and every non-empty polyhedron in standard form contain an extreme points.

$$\begin{matrix} Ax = b \\ I\cdot x \geq 0 \end{matrix} \quad \Longleftrightarrow \quad \begin{pmatrix} A \\ -A \\ I \end{pmatrix}\cdot x \geq \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}$$

<u>Example:</u>

$$\left.\begin{matrix} x_1 + x_2 \geq 1 \\ x_1 + 2x_2 \geq 0 \end{matrix}\right\} P$$

x_1, x_2, x_3 contains a line because

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in P \;\forall \lambda\in\mathbb{R}$$

2.6 Optimality of extreme points

<u>Theorem:</u> Let $P\subseteq\mathbb{R}^n$ polyhedron and $c\in\mathbb{R}^n$. If P has an extreme point and $c^T x$ is bounded from below for $x\in P$, then there exists an extreme point that is optimal.

<u>Proof:</u> We prove that for any $x\in P$ there exists an extreme point $y\in P$ with $c^T y \leq c^T x$.

Let $x\in P$ and $I := \{i \mid a_i^T x = b_i\}$. If $\{a_i \mid i\in I\}$ contains n linearly indep. vectors we are done. Otherwise $\exists\, d\in\mathbb{R}^n$ $d\neq 0$ with $a_i^T\cdot d = 0 \;\forall i\in I$. Assume that $c^T\cdot d\leq 0$ (otherwise replace d by $-d$).

<u>1. Case:</u> $c^T\cdot d < 0 \Rightarrow c^T\cdot(x+\lambda\cdot d)\to -\infty$ for $\lambda\to\infty$

Thus $\exists\, j: a_j^T\cdot d < 0$. Let $\lambda_0 := \min\limits_{j\notin I:\, a_j^T d<0} \dfrac{b_j - a_j^T x}{a_j^T d}$

$$\boxed{\text{Remark: } a_j^T(x+\lambda\cdot d) = a_j^T\cdot x + \lambda\cdot a_j^T\cdot d \geq b_j}$$

Then $x+\lambda_0 \cdot d \in P$ and $\{i \mid a_i^T(x+\lambda_0 \cdot d) = b_i\} \supseteq I \cup \{j\}$ contains more linearly independent vectors than $\{a_i : i \in I\}$. Notice that $c^T x \geq c^T(x+\lambda_0 \cdot d)$. Repeat argument with $x+\lambda_0 \cdot d$.

<u>2. Case:</u> $c^T \cdot d = 0$. $\Rightarrow$ $c^T(x+\lambda \cdot d) = c^T x \quad \forall \lambda \in \mathbb{R}$

Since P does not contain a line, we find an additional constraint in direction d or $-d$. See above, ... $\square$

<u>Corollary:</u> For any linear programming problem over a nonempty polyhedron, either the optimal cost is $-\infty$ or there exists an optimal solution.

<u>Proof:</u> Any linear program is equivalent to a linear program in standard form.

<u>Example:</u> $\min \frac{1}{x}$ s.t. $x > 0$ does not have opt. sol.

<u>Chapter 3: The Simplex Method</u>

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $\operatorname{rank}(A) = m$

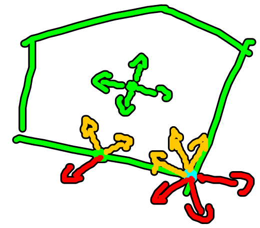
Let $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ and consider the LP

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq 0 \end{aligned}$$

<u>3.1 Optimality conditions</u>

<u>Def:</u> For $x \in P \subseteq \mathbb{R}^n$ (polyhedron), the vector $d \in \mathbb{R}^n \setminus \{0\}$ is a <u>feasible direction at x</u> if there exists $\theta > 0$

with $x + \Theta \cdot d \in P$.



Observation: Let $B = (A_{B(1)}, \dots, A_{B(m)})$ be a basis matrix. Then the basic variables $x_{B(1)}, \dots, x_{B(m)}$ in the system $Ax = b$ are uniquely determined by the non-basic variables.

$$Ax = b \iff B \cdot x_B + \sum_{j \notin B(1), \dots, B(m)} A_j \cdot x_j = b$$

$$\iff x_B = B^{-1} \cdot b - \sum_{j \notin \dots} B^{-1} \cdot A_j \cdot x_j$$