

3.1. Optimality conditions

$$\begin{aligned} \min c^T x \\ \text{s.t. } Ax = b \\ x \geq 0 \end{aligned}$$

$$A \in \mathbb{R}^{m \times n}, \text{ rank}(A) = m \\ \underbrace{(A_{B(1)}, \dots, A_{B(m)})}_{B} \text{ basis}$$

$$Ax = b \Leftrightarrow B \cdot x_B + \sum_{j \notin B(1), \dots, B(m)} A_j \cdot x_j = b$$

$$\Leftrightarrow x_B = B^{-1} \cdot b - \sum_{j \notin \dots} B^{-1} \cdot A_j \cdot x_j$$

Consider a basic feasible solution x given by $x_B = B^{-1} \cdot b \geq 0$

For some fixed $j \notin B(1), \dots, B(m)$, let $d \in \mathbb{R}^n$ be given by $d_B := -B^{-1} \cdot A_j$, $d_j := 1$ and $d_{j'} := 0$ for all $j' \neq j, B(1), \dots, B(m)$

Then $A \cdot (x + \theta d) = b \quad \forall \theta \in \mathbb{R}$

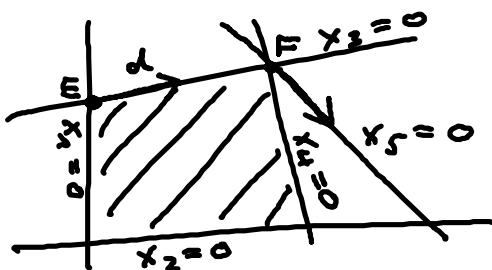
and we call d the j -th basic direction.

Question: Is d a feasible direction at x ?

Case 1: If x is a non-degenerate basic feasible solution, then $x_B > 0$ and $x + \theta \cdot d \geq 0$ for $\theta > 0$ small enough. \Rightarrow Answer is yes.

Case 2: If x is degenerate then d is not necessarily a feasible direction. E.g. if $x_{B(i)} = 0$ and $d_{B(i)} < 0$, then $x + \theta \cdot d \not\geq 0$ for $\theta > 0$.

Example: $n = 5, m = 3, n - m = 2$



in F : basic variables are x_1, x_2, x_4

How does the cost change when moving along a basic direction?

$$c^T(x + \theta \cdot d) = c^T \cdot x + \theta \cdot c^T \cdot d = c^T x + \theta \cdot \underbrace{(c_j - c_B^T \cdot B^{-1} \cdot A_j)}_{\bar{c}_j}$$

Def: For $j=1, \dots, m$ the reduced cost of variable x_j is $\bar{c}_j := c_j - c_B^T \cdot B^{-1} \cdot A_j$.

$$B^{-1} \cdot b = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Notice that $\bar{c}_{B(i)} = c_{B(i)} - c_B^T \cdot B^{-1} \cdot A_{B(i)}$
 $= c_{B(i)} - e_{B(i)} = 0$

Theorem: Let x be a basic feasible solution and \bar{c} the vector of reduced costs.

(i) IF $\bar{c} \geq 0$, then x is optimal

(ii) IF x is optimal and non-degenerate, then $\bar{c} \geq 0$.

Proof: (i) Let $y \in P$, then

$$\begin{aligned} c^T y &= c_B^T \cdot y_B + \sum_{j \in \{B(1), \dots, B(m)\}} c_j \cdot y_j \\ &= c_B^T \cdot (B^{-1} \cdot b - \sum_{j \in \dots} B^{-1} \cdot A_j y_j) + \sum_{j \in \dots} c_j y_j \\ &= c_B^T \cdot (B^{-1} \cdot b) + \sum_{j \in \dots} \underbrace{(c_j - c_B^T \cdot B^{-1} \cdot A_j)}_{\bar{c}_j} y_j \\ &\geq c_B^T \cdot (B^{-1} \cdot b) \\ &= c_B^T \cdot x_B = c^T x. \end{aligned}$$

(ii) Assume by contradiction $\bar{c}_j < 0$ for some $j \in \{B(1), \dots, B(m)\}$. Since x is a non-deg. basic feasible solution, the j -th basic direction d is a feasible direction, with $\underbrace{c^T \cdot d}_{\bar{c}_j} < 0 \Rightarrow x$ not optimal \Leftarrow \square

Def: A basis matrix B is optimal if

(i) $B^{-1} \cdot b \geq 0$ and

$$(ii) \bar{z}^T = c^T - c_B^T \cdot B^{-1} \cdot A \geq 0.$$

In particular, the associated basic solution is feasible (by (i)) and optimal (by (ii)).

3.2 Development of the simplex method

Assumption for now: Only non-degenerate basic feasible solutions.

Let x be a basic feasible solution and $\bar{c}_j < 0$ for some $j \notin B(1), \dots, B(m)$ (otherwise x is optimal)

Let d be the j -th basic direction. Then

$\bar{c}_j = c^T \cdot d < 0$ and it is desirable to go to $x + \theta \cdot d$ with $\theta^* = \max \{ \theta \geq 0 \mid x + \theta \cdot d \in P \}$

By construction of d , $A(x + \theta \cdot d) = b \quad \forall \theta \in \mathbb{R}$

$$\Rightarrow (x + \theta \cdot d \in P \Leftrightarrow x + \theta \cdot d \geq 0)$$

Case 1: $d \geq 0 \Rightarrow x + \theta \cdot d \geq 0 \quad \forall \theta \geq 0 \Rightarrow \theta^* = \infty$

\Rightarrow LP unbounded.

Case 2: $d_j < 0$ for some j : $x_j + \theta d_j \geq 0 \Leftrightarrow \theta \leq \frac{-x_j}{d_j}$

$$\Rightarrow \theta^* := \min_{\substack{j: d_j < 0 \\ i=1, \dots, m \\ d_{B(i)} < 0}} -\frac{x_j}{d_j} = \min_{\substack{i=1, \dots, m \\ d_{B(i)} < 0}} -\frac{x_{B(i)}}{d_{B(i)}} > 0$$

Let $y := x + \theta^* \cdot d$ and $l \in \{1, \dots, m\}$ with $\theta^* = -\frac{x_{B(l)}}{d_{B(l)}}$

$$\Rightarrow y_j = \theta^*, \quad y_{B(l)} = 0$$

Thus, x_j should replace $x_{B(l)}$ as a basic variable.

\rightarrow new basis matrix

$$\bar{B} = (A_{B(1)} \dots A_{B(l-1)} A_j A_{B(l+1)} \dots A_{B(m)})$$

$$= (A_{\bar{B}(1)} \dots A_{\bar{B}(m)})$$

with

$$\bar{B}(i) = \begin{cases} B(i) & \text{if } i \neq l \\ j & \text{if } i = l \end{cases}$$

Theorem: Assume that $\Theta^* < \infty$

(i) $A_{\bar{B}(1)} \dots A_{\bar{B}(m)}$ are linearly independent and \bar{B} is a basis matrix

(ii) $y = x + \Theta^* \cdot d$ is a basic feasible sol. associated with \bar{B} .

Proof: $B^{-1} \cdot \bar{B} = (e_1, e_2, \dots, e_{l-1}, -d_B e_{l+1}, \dots, e_m)$

Since $d_B < 0$, the columns of $B^{-1} \cdot \bar{B}$ are linearly indep. \Rightarrow columns of \bar{B} are lin. indep.