LP in standard form

$$\begin{array}{lll} \min \ c^T x & A \in \mathbb{R}^{m \times n} & \operatorname{rank}(A) = m \\ \text{s.t.} \quad Ax = b & c \in \mathbb{R}^n & x \in \mathbb{R}^n \\ \qquad x \geq 0 & b \in \mathbb{R}^m & \end{array}$$

$$(\overbrace{A_{B(1)}, \dots, A_{B(m)}}^{B=}) \text{ basis}$$

$\to$ basic solution $x_B := \underbrace{B^{-1} \cdot b}_{\geq 0 \text{ (feasibility)}}$, $x_j = 0$ for $j \neq B(1), \dots, B(m)$

j-th basic direction: increase non-basic variable x_j and adjust the values of $x_{B(1)}, \dots, x_{B(m)}$

$\longrightarrow$ vector d with $d_B = -B^{-1} \cdot A_j$

change of cost: $c^T \cdot d = \bar{c}_j = c_j - c_B^T \cdot B^{-1} \cdot A_j$

How far can we walk in direction d?

$$x \to x + \theta \cdot d \qquad \theta \geq 0$$

What is the maximum θ that can be chosen:

$$\theta^* := \min_{j : d_j < 0} -\frac{x_j}{d_j} = \min_{i : d_{B(i)} < 0} -\frac{x_{B(i)}}{d_{B(i)}} = -\frac{x_{B(e)}}{d_{B(e)}}$$

$\to$ new feasible solution $y := x + \theta^* \cdot d$ with $y_{B(e)} = 0$ and $y_j := \theta^*$ and $y_i = 0$ for all $i \neq B(1), \dots, B(m), j$

$\to$ new basis $(A_{B(1)} \cdots A_{B(e-1)} \, A_j \, A_{B(e+1)} \cdots A_{B(m)}) = \bar{B}$

$$B^{-1} \cdot (\quad) = \begin{array}{c} \begin{pmatrix} 1 & & \vert & & \\ & \ddots & & & \\ & & d_B & & \\ & & & \ddots & \\ & & \vert & & 1 \end{pmatrix} \\ e \end{array}$$

y is the basic feasible solution corresponding to the new basis $\bar{B}$

Proof: By construction, $y_j = 0$ for $j \neq \bar{B}(1), \dots, \bar{B}(m)$ and $A \cdot y = b \to y_{\bar{B}} = \bar{B}^{-1} \cdot b$. $\square$

Notice that $c^T y = c^T x + \theta^* \cdot c^T d = c^T x + \underbrace{\theta^*}_{>0} \cdot \underbrace{\bar{c}_j}_{<0} < c^T x$.

$Ax = b$ ← m active constraints

$x \geq 0$ ← $n-m$ active non-negativity constraints

Example: $n = 6$, $m = 4$ → $n - m = 2$

Initial basic feasible solution. Basic variables are x_1, x_2, x_3, x_6

Basic direction corresponding to x_4 is g.

" " " " x_5 is f.

Choosing x_4 to enter the basis, x_6 must leave the basis. → new basis with basic variables x_1, x_2, x_3, x_4

Basic direction corresponding to x_6 is $-g$

" " " " x_5 is h.

The reduced cost of x_6 is $\bar{c}_6 > 0$

" " " " x_5 is $\bar{c}_5 < 0$!

$$\begin{pmatrix} 1 & & & \vdots & & & \\ & \ddots & & \vdots & & & \\ & & 1 & \vdots & & & \\ & & & u_\ell & & & \\ & & & \vdots & 1 & & \\ & & & \vdots & & \ddots & \\ & & & \vdots & & & 1 \end{pmatrix}$$