

Chapter 4: Duality Theory

4.1 Motivation

$$\begin{array}{ll} \min c^T x & A \in \mathbb{R}^{m \times n} \\ \text{s.t. } Ax \geq b & \\ & x \geq 0 \end{array}$$

Try to derive lower bounds on the value of an optimum solution. Use the information that $Ax \geq b$ for any feasible solution x .

For $p \in \mathbb{R}^m$ with $p \geq 0$: $Ax \geq b \Rightarrow (p^T A) \cdot x \geq p^T b$

If $c^T \geq p^T A$, then $c^T x \geq (p^T A) \cdot x \geq p^T b$ for all feasible solutions to our LP.

How to find the best (largest) lower bound in this way?

$$\begin{array}{ll} \max p^T b & \leftrightarrow \max s^T \cdot p \\ \text{s.t. } p^T A \leq c^T & \text{s.t. } A^T \cdot p \leq c \\ & p \geq 0 \end{array}$$

This LP is the dual linear program of our initial LP.

More general:

4.2 The dual problem

min $c^T x$

s.t. $a_i^T \cdot x \geq b_i$ for $i \in M_1$

$a_i^T \cdot x \leq b_i$ for $i \in M_2$

$a_i^T x = b_i$ for $i \in M_3$

$x_j \geq 0$ for $j \in N_1$

$x_j \leq 0$ for $j \in N_2$

x_j free for $j \in N_3$

Lower bound on $c^T x$ can be obtained by choosing $p_i, i \in M_1 \cup M_2 \cup M_3$ with:

s.t. $p_i \geq 0$ for $i \in M_1$
 $p_i \leq 0$ for $i \in M_2$
 p_i free for $i \in M_3$

$\boxed{\max p^T \cdot b}$

and: $p^T \cdot A_j \leq c_j$ for $j \in N_1$

$p^T \cdot A_j \geq c_j$ for $j \in N_2$

$p^T \cdot A_j = c_j$ for $j \in N_3$

This linear program is the dual LP of the primal linear program we started with.

Example:

min $c^T x$

s.t. $Ax \geq b$

primal LP

max $p^T b$

$p^T A = c^T$

$p \geq 0$

Theorem: The dual of the dual LP is the

primal LP.

Proof: (for special case only)

Consider primal LP:

$$\begin{aligned} \min c^T x \\ \text{s.t. } Ax = b \\ x \geq 0 \end{aligned}$$

dualize \rightarrow

$$\begin{aligned} \max p^T b \\ \text{s.t. } p^T A \leq c^T \\ p \text{ free} \end{aligned}$$

equivalent \rightarrow

$$\begin{aligned} \min (-b^T) \cdot p \\ \text{s.t. } A^T \cdot p \leq c \end{aligned}$$

dualize \rightarrow

$$\begin{aligned} \max y^T \cdot c \\ \text{s.t. } y^T \cdot A^T = -b^T \\ y \leq 0 \end{aligned}$$

equivalent \rightarrow

$$\begin{aligned} \min c^T \cdot (-y) \\ \text{s.t. } A \cdot (-y) = b \\ (-y) \geq 0 \end{aligned}$$

$$\begin{aligned} \rightarrow \min c^T z \\ \text{s.t. } A \cdot z = b \\ z \geq 0 \end{aligned}$$

□