3.7 Computational efficiency of the simplex method

Observation

The computational efficiency of the simplex method is determined by

1. the computational effort of each iteration;
2. the number of iterations.

Question

How many iterations are needed in the worst case?

Idea for negative answer (lower bound)

Describe

- a polyhedron with an exponential number of vertices;
- a path that visits all vertices and always moves from a vertex to an adjacent one that has lower costs.

3.7 Computational efficiency of the simplex method

Observation

The computational efficiency of the simplex method is determined by

1. the computational effort of each iteration;
2. the number of iterations.

Question

How many iterations are needed in the worst case?

Idea for negative answer (lower bound)

Describe

- a polyhedron with an exponential number of vertices;
- a path that visits all vertices and always moves from a vertex to an adjacent one that has lower costs.

3.7 Computational efficiency of the simplex method

Observation

The computational efficiency of the simplex method is determined by

1. the computational effort of each iteration;
2. the number of iterations.

Question

How many iterations are needed in the worst case?

Idea for negative answer (lower bound)

Describe

- a polyhedron with an exponential number of vertices;
- a path that visits all vertices and always moves from a vertex to an adjacent one that has lower costs.

Computational efficiency of the simplex method

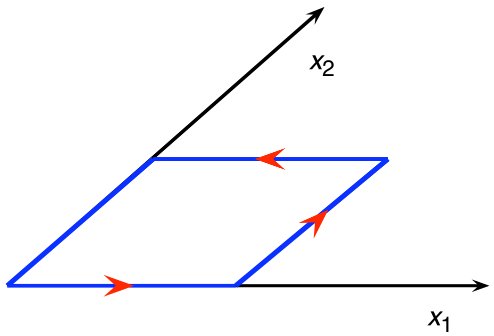
Unit cube

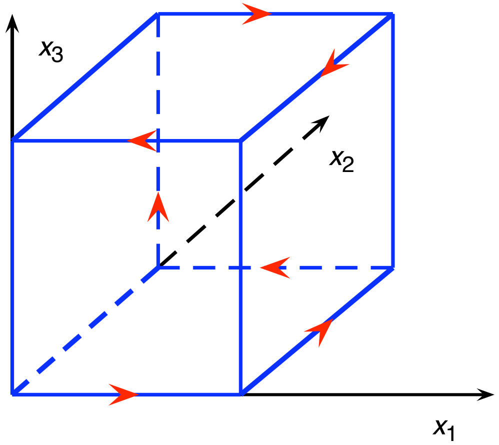
Consider the unit cube in $\mathbb{R}^n$, defined by the constraints

$$0 \leq x_i \leq 1, \quad i = 1, \ldots, n$$

The unit cube has

- 2^n vertices;
- a *spanning path*, i.e. a path traveling the edges of the cube visiting each vertex exactly once.





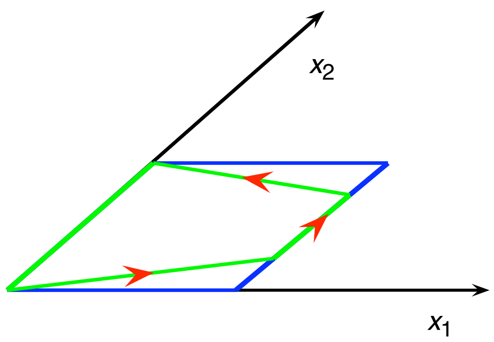
Computational efficiency of the simplex method

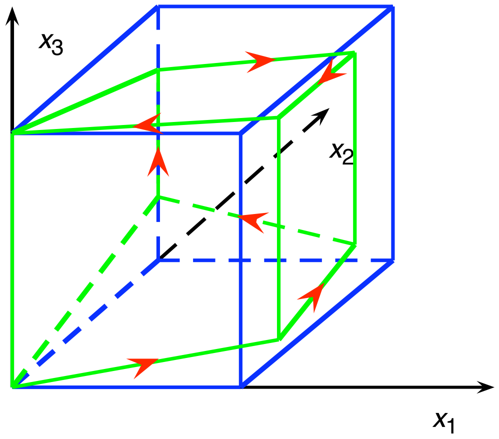
Klee-Minty cube

Consider a perturbation of the unit cube in $\mathbb{R}^n$, defined by the constraints

$$0 \le x_1 \le 1,$$
$$\epsilon x_{i-1} \le x_i \le 1 - \epsilon x_{i-1}, \quad i = 2, \ldots, n$$

for some $\epsilon \in (0, 1/2)$.





Computational efficiency of the simplex method

Klee-Minty cube

$$0 \le x_1 \le 1,$$
$$\epsilon x_{i-1} \le x_i \le 1 - \epsilon x_{i-1}, \quad i = 2, \ldots, n, \epsilon \in (0, 1/2)$$

Theorem

Consider the linear programming problem of minimizing $-x_n$ subject to the constraints above. Then:

1. *The feasible set has 2^n vertices.*
2. *The vertices can be ordered so that each one is adjacent to and has lower cost than the previous one.*
3. *There exists a pivoting rule under which the simplex method requires $2^n - 1$ changes of basis before it terminates.*

Computational efficiency of the simplex method

Klee-Minty cube

$$0 \le x_1 \le 1,$$
$$\epsilon x_{i-1} \le x_i \le 1 - \epsilon x_{i-1}, \quad i = 2, \dots, n, \epsilon \in (0, 1/2)$$

Theorem

Consider the linear programming problem of minimizing $-x_n$ subject to the constraints above. Then:

1. *The feasible set has 2^n vertices.*
2. *The vertices can be ordered so that each one is adjacent to and has lower cost than the previous one.*
3. *There exists a pivoting rule under which the simplex method requires $2^n - 1$ changes of basis before it terminates.*

The diameter of polyhedra

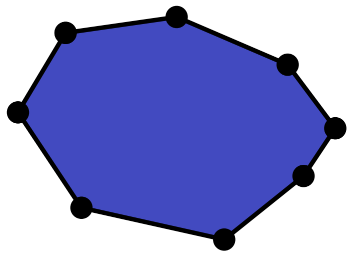
Definition

- The *distance* $d(x, y)$ between two vertices x, y is the minimum number of edges required to reach y starting from x.
- The *diameter* $D(P)$ of polyhedron P is the maximum $d(x, y)$ over all pairs of vertices (x, y).
- $\Delta(n, m)$ is the maximum $D(P)$ over *all bounded* polyhedra in $\mathbb{R}^n$ that are represented in terms of m inequality constraints.
- $\Delta_u(n, m)$ is the maximum $D(P)$ over *all* polyhedra in $\mathbb{R}^n$ that are represented in terms of m inequality constraints.

The diameter of polyhedra

Definition

- The *distance* $d(x, y)$ between two vertices x, y is the minimum number of edges required to reach y starting from x.
- The *diameter* $D(P)$ of polyhedron P is the maximum $d(x, y)$ over all pairs of vertices (x, y).
- $\Delta(n, m)$ is the maximum $D(P)$ over *all bounded* polyhedra in $\mathbb{R}^n$ that are represented in terms of m inequality constraints.
- $\Delta_u(n, m)$ is the maximum $D(P)$ over *all* polyhedra in $\mathbb{R}^n$ that are represented in terms of m inequality constraints.

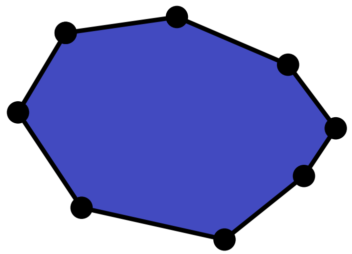


$$\Delta(2, 8) = \left\lfloor \frac{8}{2} \right\rfloor = 4$$

The diameter of polyhedra

Definition

- The *distance* $d(x, y)$ between two vertices x, y is the minimum number of edges required to reach y starting from x.
- The *diameter* $D(P)$ of polyhedron P is the maximum $d(x, y)$ over all pairs of vertices (x, y).
- $\Delta(n, m)$ is the maximum $D(P)$ over *all bounded* polyhedra in $\mathbb{R}^n$ that are represented in terms of m inequality constraints.
- $\Delta_u(n, m)$ is the maximum $D(P)$ over *all* polyhedra in $\mathbb{R}^n$ that are represented in terms of m inequality constraints.



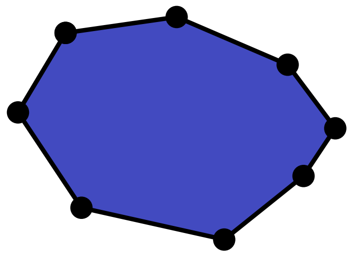
$$\Delta(2, 8) = \left\lfloor \frac{8}{2} \right\rfloor = 4$$

$$\Delta(2, m) = \left\lfloor \frac{m}{2} \right\rfloor$$

The diameter of polyhedra

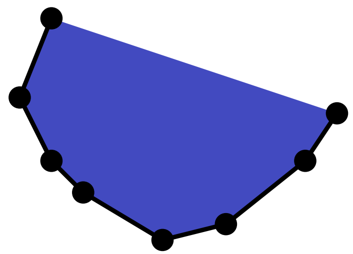
Definition

- The *distance* $d(x, y)$ between two vertices x, y is the minimum number of edges required to reach y starting from x.
- The *diameter* $D(P)$ of polyhedron P is the maximum $d(x, y)$ over all pairs of vertices (x, y).
- $\Delta(n, m)$ is the maximum $D(P)$ over *all bounded* polyhedra in $\mathbb{R}^n$ that are represented in terms of m inequality constraints.
- $\Delta_u(n, m)$ is the maximum $D(P)$ over *all* polyhedra in $\mathbb{R}^n$ that are represented in terms of m inequality constraints.



$\Delta(2, 8) = \left\lfloor \frac{8}{2} \right\rfloor = 4$

$\Delta(2, m) = \left\lfloor \frac{m}{2} \right\rfloor$

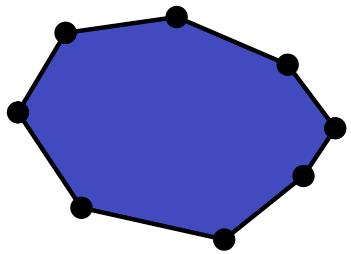


$\Delta_u(2, 8) = 8 - 2 = 6$

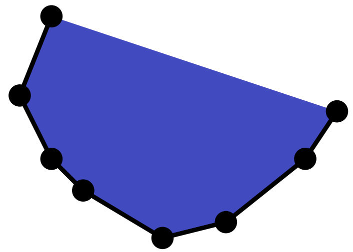
The diameter of polyhedra

Definition

- The *distance* $d(x, y)$ between two vertices x, y is the minimum number of edges required to reach y starting from x.
- The *diameter* $D(P)$ of polyhedron P is the maximum $d(x, y)$ over all pairs of vertices (x, y).
- $\Delta(n, m)$ is the maximum $D(P)$ over *all bounded* polyhedra in $\mathbb{R}^n$ that are represented in terms of m inequality constraints.
- $\Delta_u(n, m)$ is the maximum $D(P)$ over *all* polyhedra in $\mathbb{R}^n$ that are represented in terms of m inequality constraints.



$$\Delta(2, 8) = \left\lfloor \frac{8}{2} \right\rfloor = 4$$
$$\Delta(2, m) = \left\lfloor \frac{m}{2} \right\rfloor$$



$$\Delta_u(2, 8) = 8 - 2 = 6$$
$$\Delta_u(2, m) = m - 2$$

The Hirsch Conjecture

Observation

The diameter of the feasible set in a linear programming problem is a lower bound on the number of steps required by the simplex method, no matter which pivoting rule is being used.

Hirsch Conjecture

$$\Delta(n, m) \leq m - n$$

Known bounds

- Lower bounds: $\Delta_u(n, m) \geq m - n + \left\lfloor \frac{n}{5} \right\rfloor$
- Upper bounds:

$$\Delta(n, m) \leq \Delta_u(n, m) < m^{1+\log_2 n} = (2n)^{\log_2 m}$$

The Hirsch Conjecture

Observation

The diameter of the feasible set in a linear programming problem is a lower bound on the number of steps required by the simplex method, no matter which pivoting rule is being used.

Hirsch Conjecture

$$\Delta(n, m) \leq m - n$$

Known bounds

- Lower bounds: $\Delta_u(n, m) \geq m - n + \left\lfloor \frac{n}{5} \right\rfloor$
- Upper bounds:

$$\Delta(n, m) \leq \Delta_u(n, m) < m^{1+\log_2 n} = (2n)^{\log_2 m}$$

The Hirsch Conjecture

Observation

The diameter of the feasible set in a linear programming problem is a lower bound on the number of steps required by the simplex method, no matter which pivoting rule is being used.

Hirsch Conjecture

$$\Delta(n, m) \leq m - n$$

Known bounds

- Lower bounds: $\Delta_u(n, m) \geq m - n + \left\lfloor \frac{n}{5} \right\rfloor$
- Upper bounds:

$$\Delta(n, m) \leq \Delta_u(n, m) < m^{1+\log_2 n} = (2n)^{\log_2 m}$$

The Hirsch Conjecture

Observation

The diameter of the feasible set in a linear programming problem is a lower bound on the number of steps required by the simplex method, no matter which pivoting rule is being used.

Hirsch Conjecture

$$\Delta(n, m) \leq m - n$$

Known bounds

- Lower bounds: $\Delta_u(n, m) \geq m - n + \left\lfloor \frac{n}{5} \right\rfloor$
- Upper bounds:

$$\Delta(n, m) \leq \Delta_u(n, m) < m^{1+\log_2 n} = (2n)^{\log_2 m}$$

Average case behavior of the simplex method

Remark

- Despite the exponential lower bounds on the worst case behavior of the simplex method (Klee-Minty cubes etc.), the simplex method usually behaves well in practice.
- The number of iterations is "typically" $O(m)$.
- There have been several attempts to explain this phenomenon from more a theoretical point of view.
- These results say that "on average" the number of iterations is $O(\cdot)$ (usually polynomial).
- One main difficulty is to come up with a meaningful and, at the same time, manageable definition of the term "on average".