LP duality

$$\begin{array}{lll}
\min c^T x & \max p^T \cdot b & \\
a_i^T \cdot x \geq b_i & p_i \geq 0 & i \in M_1 \\
a_i^T x \leq b_i & p_i \leq 0 & i \in M_2 \\
a_i^T x = b_i & p_i \text{ free} & i \in M_3 \\
x_j \geq 0 & p^T \cdot A_j \leq c_j & j \in N_1 \\
x_j \leq 0 & p^T \cdot A_j \geq c_j & j \in N_2 \\
x_j \text{ free} & p^T \cdot A_j = c_j & j \in N_3
\end{array}$$

<u>Theorem:</u> The dual of the dual LP is the primal LP.

<u>Theorem:</u> Let Π_1 and Π_2 be two LPs where Π_2 has been obtained from Π_1 by (several) transformations of the following type

(i) Replace a free variable by difference of two non-negative variables.

(ii) Introducing a slack variable in order to replace an inequality constr. by equal. constr.

(iii) If some row of a feasible equality system is a linear combination of the other rows, eliminate this row.

Then the dual of Π_1 is equivalent to the dual of Π_2.

Sketch of proof:

$$\begin{array}{lcl} \min\ c^T x & \xrightarrow{\text{dualize}} & \max\ p^T \cdot b \\ \text{s.t. } Ax \geq b & & \text{s.t. } p^T \cdot A = c^T \\ \quad x \text{ free} & & \quad p \geq 0 \end{array}$$

(ii) $\updownarrow$ equivalent $\qquad\qquad$ $\updownarrow$ equivalent

$$\begin{array}{lcl} \min\ c^T x & \xrightarrow{\text{dualize}} & \max\ p^T \cdot b \\ \text{s.t. } Ax - Is = b & & \text{s.t. } p^T \cdot A = c^T \\ \quad x \text{ free} & & \quad p^T \cdot (-I) \leq 0 \\ \quad s \geq 0 & & \quad p \text{ free} \end{array}$$

(i) $\updownarrow$ equivalent $\qquad\qquad$ $\updownarrow$ equivalent

$$\begin{array}{lcl} \min\ c^T x^+ - c^T x^- & \xrightarrow{\text{dualize}} & \max\ p^T \cdot b \\ \text{s.t. } A \cdot x^+ - A \cdot x^- \geq b & & \text{s.t. } p^T \cdot A \leq c^T \\ \quad x^+, x^- \geq 0 & & \quad p^T \cdot (-A) \leq -c^T \\ & & \quad p \geq 0 \end{array}$$

(iii) Let $a_1^T, \dots, a_m^T$ rows of A and suppose that

$$a_m^T = \sum_{i=1}^{m-1} \gamma_i \cdot a_i^T \quad \text{and} \quad b_m = \sum_{i=1}^{m-1} \gamma_i \cdot b_i$$

Then row m is redundant and can be eliminated. The dual constraints

$$\sum_{i=1}^{m} p_i \cdot a_i^T \leq c^T \quad \text{can be rewritten}$$

as $$\sum_{i=1}^{m-1} (p_i + \gamma_i \cdot p_m) \cdot a_i^T \leq c^T$$

and the dual cost is

$$\sum_{i=1}^{m} p_i \cdot b_i = \sum_{i=1}^{m-1} (p_i + \gamma_i \cdot p_m) \cdot b_i$$

Let $q_i := p_i + \gamma_i \cdot p_m$

$$\text{dual LP} \overset{\text{equiv.}}{\longleftrightarrow} \quad \max \sum_{i=1}^{m-1} q_i \cdot b_i$$

$$\text{s.t.} \quad \sum_{i=1}^{m-1} q_i \cdot a_i^T \leq c^T \qquad \square$$

4.3 Duality Theorem

Theorem (weak duality):

If x feasible sol. to primal LP and
p " " " dual ", then

$$c^T x \geq p^T \cdot b.$$

Proof: Consider a primal-dual pair of LPs.

$$\begin{array}{llll} \min & c^T x & & \\ \text{s.t.} & Ax \geq b & (1) & \\ & x \geq 0 & & \end{array} \qquad \begin{array}{llll} \max & p^T \cdot b & & \\ \text{s.t.} & p^T A \leq c^T & (2) & \\ & p \geq 0 & & \end{array}$$

$$c^T x \underset{\substack{\uparrow \\ p \text{ fulfills } (2) \\ x \geq 0}}{\geq} (p^T \cdot A) \cdot x = p^T \cdot (A \cdot x) \underset{\substack{\uparrow \\ x \text{ fulfills } (1) \\ p \geq 0}}{\geq} p^T \cdot b \qquad \square$$

Corollary: (i) If primal LP is unbounded (i.e. opt. cost $= -\infty$), then the dual LP is infeasible.

(ii) If dual LP is unbounded, then primal LP is infeasible.

<u>Corollary:</u> If x, p are feasible sol. to the primal and dual LP, respectively, and if $c^T x = p^T b$, then x and p are optimal solutions.

<u>Theorem (strong duality):</u>

If an LP has an optimal solution, so does its dual and the optimal costs are equal.

<u>Proof:</u> Consider LP in standard form with linearly indep. rows of A:

$$\begin{array}{llcll} \min & c^T x & \qquad & \max & p^T b \\ \text{s.t.} & Ax = b & & \text{s.t.} & p^T A \le c^T \\ & x \ge 0 & & & \end{array}$$

The simplex method yields opt. basis B with

$$\bar{c}^T = c^T - c_B^T \cdot B^{-1} \cdot A \ge 0$$

and a corresp. opt. basic feas. sol. x with

$$x_B = B^{-1} \cdot b.$$

Let $p^T := c_B^T \cdot B^{-1}$. Then

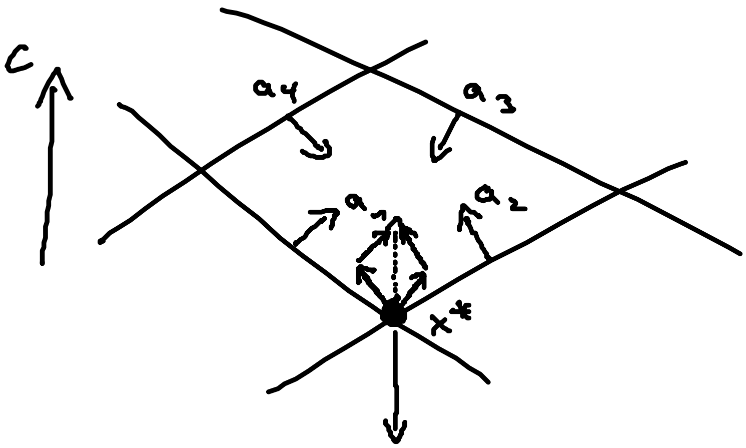
$$p^T \cdot A = c_B^T \cdot B^{-1} \cdot A \le c^T$$

and $p^T \cdot b = c_B^T \cdot (B^{-1} \cdot b) = c_B^T \cdot x_B = c^T \cdot x.$ $\square$

<u>Interpretation of duality:</u>

$$\min c^T x \qquad \text{s.t. } Ax \geq b$$

$$\max p^T b \qquad \text{s.t. } p^T \cdot A = c^T, \quad p \geq 0$$



$$c^T x^* = p^{*T} \cdot b$$

$$c^T \cdot x^* = p^{*T} A \cdot x^* \geq p^{*T} \cdot b$$

$$\Rightarrow p_i = 0 \text{ if } a_i^T \cdot x^* > b_i$$

<u>Complementary slackness</u>:

<u>Theorem:</u> Let x, p feasible primal and dual solutions, then x, p are both optimal if and only if:

$$u_i := p_i \cdot (a_i^T x - b_i) = 0 \quad \forall i$$

$$v_j := (c_j - p^T \cdot A_j) \cdot x_j = 0 \quad \forall j$$

<u>Proof:</u> Feasibility $\Rightarrow$ $u_i, v_j \geq 0 \quad \forall i, j$

$$c^T x \geq (p^T \cdot A) \cdot x = \underbrace{p^T \cdot (Ax)}_{(*)} \geq \underbrace{p^T b}_{(**)}$$

similar for v_j's.

$$(*) - (**) = \sum u_i$$

□