<u>Weak duality:</u> x, p feasible $\Rightarrow c^T x \geq p^T b$

<u>Strong duality:</u> If primal LP has opt. sol., then also dual LP has opt. sol. and optimal costs are equal.

<u>Complementary Slackness:</u> x, p feasible

x, p both optimal $\Leftrightarrow$ $p_i \cdot (a_i^T x - b_i) = 0 \quad \forall i$

$x_j \cdot (c_j - p^T A_j) = 0 \quad \forall j$

All different possibilities for primal-dual pair of LPs:

dual \ primal	finite opt.	unbounded	infeasible
finite opt.	✓	—	—
unbounded	—	—	✓
infeasible	—	✓	✓

Example where primal LP and dual LP are infeas.

$$\begin{aligned} \min \ & x_1 + 2x_2 \\ \text{s.t.} \ & x_1 + x_2 = 1 \\ & 2x_1 + 2x_2 = 3 \end{aligned} \qquad \begin{aligned} \max \ & p_1 + 3 \cdot p_2 \\ \text{s.t.} \ & p_1 + 2p_2 = 1 \\ & p_1 + 2p_2 = 2 \end{aligned}$$

<u>A geometric view</u>

$$\begin{aligned} \min \ & c^T x \\ \text{s.t.} \ & Ax \geq b \end{aligned} \qquad \begin{aligned} \max \ & p^T \cdot b \\ \text{s.t.} \ & p^T \cdot A = c^T \\ & p \geq 0 \end{aligned} \qquad \begin{aligned} & A \in \mathbb{R}^{m \times n} \\ & \operatorname{rank}(A) = n \end{aligned}$$

Let $I \subseteq \{1, \dots, m\}$ with $a_i, i \in I$, linearly independent.

$|I| = n$

$\rightarrow a_i^T x = b_i,\ i \in I$, has unique solution x^I (basic solution)

Let $p \in \mathbb{R}^m$ (dual vector). x^I, p are optimal if

(i) $a_i^T \cdot x^I \geq b_i \ \forall i$ (primal feasibility)

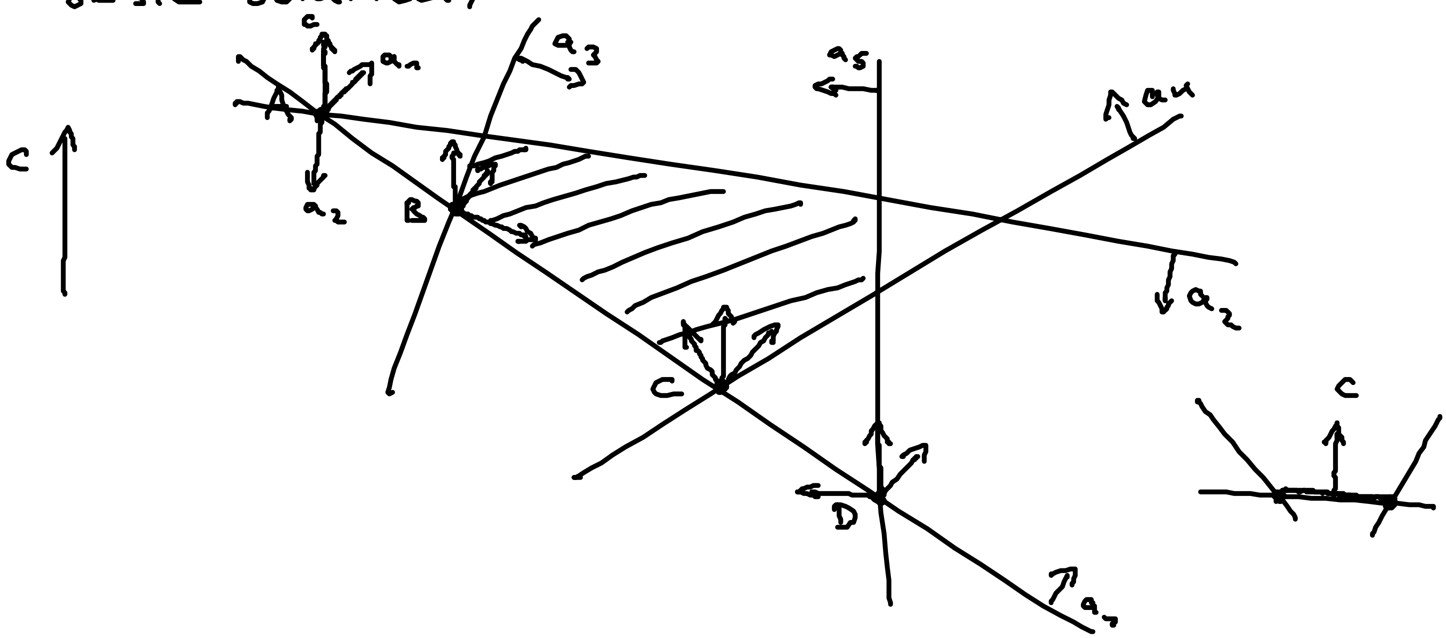
(ii) $p_i = 0 \ \forall i \notin I$ (compl. slackness)

(iii) $\sum_{i=1}^{m} p_i \cdot a_i = c$ (dual feasibility)

(iv) $p \geq 0$ (" ")

(ii) & (iii) $\Rightarrow \sum_{i \in I} p_i \cdot a_i = c \rightarrow$ unique solution p^I

($a_i, i \in I$, form basis for dual problem, p^I is corresp. basic solution)



A is primal and dual infeasible

B is primal feasible, dual infeasible

C is primal feasible, dual feasible

D is primal infeasible, dual feasible.

4.4 Optimal dual variables as marginal costs

$$\begin{array}{ll} \min c^T x & \max p^T \cdot b \\ \text{s.t. } Ax = b & \text{s.t. } p^T \cdot A \leq c^T \\ \quad x \geq 0 & \end{array}$$

Let x^* be optimal basic feas. sol. for primal LP with basis B, i.e. $x_B^* = B^{-1} \cdot b$. Assume that x^* non-deg., $x_B^* > 0$. Replace b by $b+d$. For small d, B remains optimal basis, since:

$$B^{-1} \cdot (b+d) = B^{-1} \cdot b + B^{-1} \cdot d \geq 0 \quad \text{(feasibility)}$$

and $\bar{c}^T = c^T - c_B^T \cdot B^{-1} \cdot A \geq 0$ (optimality)

Optimal cost of the perturbed problem is:

$$c_B^T \cdot B^{-1} \cdot (b+d) = c_B^T \cdot x_B^* + \underbrace{(c_B^T \cdot B^{-1})}_{p} \cdot d$$

$\longrightarrow$ p_i is marginal cost per unit increase of b_i.

4.5 The dual simplex method

$$\begin{array}{ll} \min c^T x & \max p^T b \\ \text{s.t. } Ax = b & \text{s.t. } p^T A \leq c^T \\ \quad x \geq 0 & \end{array}$$

basis B $\longrightarrow$ dual variables $p^T = c_B^T \cdot B^{-1}$

primal opt. cond. $\longleftrightarrow$ dual feasibility

$$\begin{array}{c|ccc} -c_B^T x_B & \bar{c}_1 & \cdots & \bar{c}_n \\ \hline x_{B(1)} & | & & | \\ \vdots & B^{-1} A_1 & \cdots & B^{-1} \cdot A_n \\ x_{B(m)} & | & & | \end{array}$$

Assume that $\bar{c}_i \geq 0 \ \forall i$, i.e. $p^T A = c_B^T \cdot B^{-1} \cdot A \leq c^T$ and p is feasible with cost

$$p^T \cdot b = c_B^T \cdot B^{-1} \cdot b = c_B^T \cdot x_B = c^T \cdot x$$

If $B^{-1} \cdot b \geq 0$, then x, p are optimal.

Assume that $x_{B(\ell)} < 0$ and consider ℓ-th row

$$(x_{B(\ell)} \; v_1 \; \ldots \; v_n) \qquad \text{(pivot row)}$$

I: Let $j \in \{1, \ldots, n\}$ with $v_j < 0$ and

$$\frac{\bar{c}_j}{|v_j|} = \min_{\{i \mid v_i < 0\}} \frac{\bar{c}_i}{|v_i|}$$

Then, choosing v_j as the pivot element and performing one iteration of the simplex method yields a new basis B' with $c_{B'}^T \cdot B'^{-1} \cdot A \leq c^T$ and

$$p'^T \cdot b \geq p^T b \quad (\text{with} \;„>" \text{ if } \bar{c}_j > 0).$$

II: If $v_i \geq 0 \; \forall i \in \{1, \ldots, n\}$, then the dual LP is unbounded and the primal LP is infeasible.

$\longrightarrow$ dual simplex method.

<u>Remark:</u> Dual simplex method terminates if a lexicographic pivoting rule is applied.

<u>When to apply the dual simplex method?</u>

If basic feasible solution of dual LP is readily available (but not for primal LP).

<u>Example:</u> We solved primal LP and only afterwards we realize that the right-hand side vector b is not correct $\longrightarrow$ use iterations of dual

simplex method to reoptimize for correct r.h.s.
(much faster than resolving LP from scratch.)

<u>Example:</u>

$$\begin{aligned} \min\ & x_1 + x_2 \\ \text{s.t.}\ & x_1 + 2x_2 - x_3 = 2 \\ & x_1 \phantom{{}+ 2x_2} - x_4 = 1 \\ & x_i \geq 0 \quad i = 1, \dots, 4 \end{aligned}$$