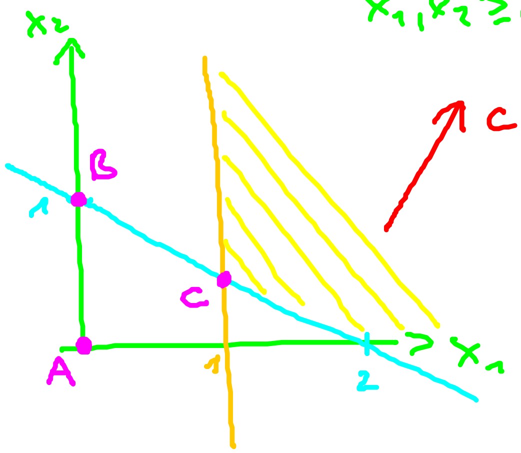
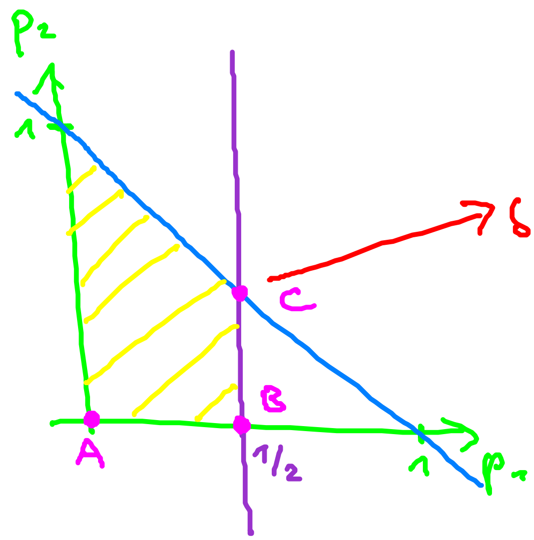
Example:

$$\begin{array}{llr} \min & x_1 + x_2 & \\ \text{s.t.} & x_1 + 2x_2 - x_3 & = 2 \\ & x_1 - x_4 & = 1 \\ & x_1, \dots, x_4 \geq 0 & \end{array} \qquad \begin{array}{ll} \max & 2p_1 + p_2 \\ \text{s.t.} & \boxed{p_1 + p_2 \leq 1} \\ & \boxed{2p_1 \leq 1} \\ & p_1, p_2 \geq 0 \end{array}$$

$\Updownarrow$ equivalent

$$\begin{array}{ll} \min & x_1 + x_2 \\ \text{s.t.} & \boxed{x_1 + 2x_2 \geq 2} \\ & \boxed{x_1 \geq 1} \\ & x_1, x_2 \geq 0 \end{array}$$





A
$$\begin{array}{c|cccc} & 0 & 1 & 1 & 0 & 0 \\ \hline x_3 & -2 & -1 & -2 & 1 & 0 \\ x_4 & -1 & -1 & 0 & 0 & 1 \end{array}$$

$\longrightarrow$ B
$$\begin{array}{c|cccc} & -1 & 1/2 & 0 & 1/2 & 0 \\ \hline x_2 & 1 & 1/2 & 1 & -1/2 & 0 \\ x_4 & -1 & -1 & 0 & 0 & 1 \end{array}$$

C $\longrightarrow$
$$\begin{array}{c|cccc} & -3/2 & 0 & 0 & 1/2 & 1/2 \\ \hline x_2 & 1/2 & 0 & 1 & -1/2 & 1/2 \\ x_1 & 1 & 1 & 0 & 0 & -1 \end{array}$$

Conclusions: Consider primal LP in standard form and its dual:

(i) Every basis B determines a basic solution to the primal ($x_B = B^{-1} \cdot b$, $x_i = 0$ for $i \notin B(1), \dots, B(m)$) but also to the dual: $p^T = c_B^T \cdot B^{-1}$.

(ii) p is feasible $\Leftrightarrow$ $\bar{c} \geq 0$

(iii) Reduced cost $\bar{c}_i = 0$ corresponds to active dual constraint.

(iv) p is degenerate $\Leftrightarrow$ $\bar{c}_i = 0$ for some non-basic variable x_i

4.6 Farkas' lemma

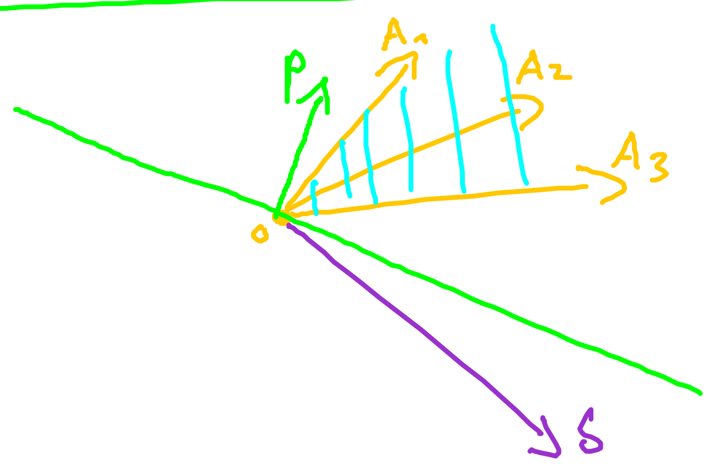
Theorem (Farkas' lemma)

For $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ exactly one of the following two alternatives holds:

(i) $\exists x \geq 0 : Ax = b$

(ii) $\exists p : p^T A \geq 0$ and $p^T b < 0$

Geometric illustration:



Proof: $(i) \Rightarrow \neg(ii)$:

Let $p \in \mathbb{R}^m$ with $p^T A \geq 0$. Then

$$p^T \cdot b = \underbrace{(p^T \cdot A)}_{\geq 0} \underbrace{x}_{\geq 0} \geq 0$$

$\neg(i) \Rightarrow (ii)$:

$$\max\ 0^T \cdot x \quad \text{s.t. } Ax = b,\ x \geq 0$$

infeasible. Dual:

$$\min\ p^T b \quad \text{s.t. } p^T A \geq 0$$

unbounded or infeasible

$p = 0$ feasible solution to dual LP

$\Rightarrow$ dual is unbounded.

$\Rightarrow \exists p : p^T A \geq 0$ and $p^T b < 0$. $\square$

<u>Theorem:</u> Let $Ax \leq b$ feasible and $d \in \mathbb{R}$. $c \in \mathbb{R}^n$

The following are equivalent:

(i) $\forall x : Ax \leq b \Rightarrow c^T x \leq d$

(ii) $\exists p \geq 0 : p^T A = c^T$ and $p^T \cdot b \leq d$

<u>Proof:</u> Consider primal-dual pair

$$\begin{array}{ll} \max\ c^T x & \min\ p^T \cdot b \\ \text{s.t. } Ax \leq b & \text{s.t. } p^T A = c^T \\ & \qquad p \geq 0 \end{array}$$

(i) $\Leftrightarrow$ opt cost for primal LP $\leq d$

(ii) $\Leftrightarrow$ " " " dual " $\leq d$

$\square$

<u>4.8 Cones and extreme rays</u>

<u>Def:</u> A set $C \subseteq \mathbb{R}^n$ is a cone if $\lambda \cdot x \in C$ for all $x \in C$ and $\lambda \geq 0$.

Notice that $0 \in C$ for every nonempty cone C.

<u>Example:</u> A polyhedron of the form

$$P = \{x \mid Ax \geq 0\}$$

is a <u>polyhedral</u> cone.

Notice that $x \in P$ with $x \neq 0$ is not a vertex of P:

$$x = \frac{1}{2} \cdot \Big(\underbrace{\tfrac{1}{2} x}_{\in P} + \underbrace{\tfrac{3}{2} x}_{\in P} \Big) \quad \text{(convex combination)}$$

$\Rightarrow 0$ is the only possible vertex of a polyhedral cone.

If $0 \in P$ is a vertex, then P is <u>pointed</u>.

<u>Theorem:</u> Let $C = \{x \mid a_i^T x \geq 0, i = 1, \ldots, m\}$. The following are equivalent:

(i) 0 is an extreme point of C

(ii) C does not contain a line

(iii) There exist n linearly independent vectors in $\{a_i \mid i = 1, \ldots, m\}$.

<u>Proof:</u> See Sect. 2.5. $\square$