

4.8 Cones and extreme rays

$C \subseteq \mathbb{R}^n$ cone if $\forall x \in C, \lambda \geq 0 : \lambda \cdot x \in C$
 $\Rightarrow 0 \in C$ if $C \neq \emptyset$

$C = \{x \in \mathbb{R}^n \mid A \cdot x \geq 0\}$ polyhedral cone

0 is the only possible vertex of a polyhedral cone.

If 0 is vertex of C , then C is pointed.

Theorem: Equivalent:

- (i) 0 is extreme point of C
- (ii) C does not contain a line
- (iii) A has n linearly indep. rows.

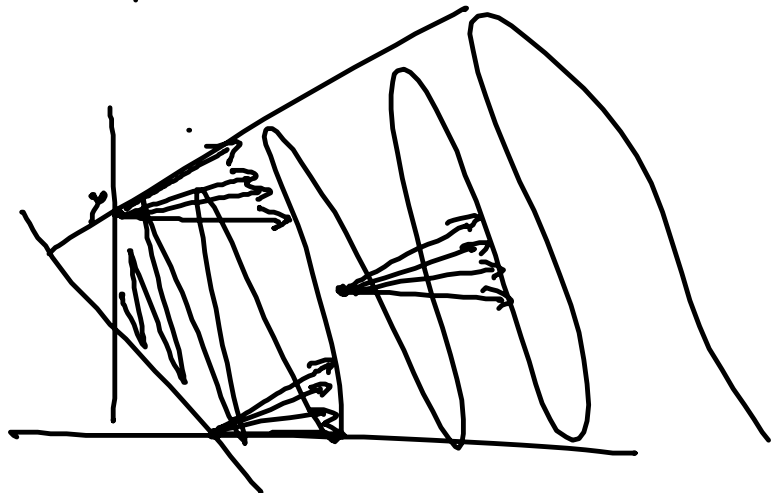
Let $P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$ and $y \in P$. The recession cone
(at y) of P is $\{d \in \mathbb{R}^n \mid \underbrace{A \cdot (y + \lambda \cdot d)}_{y + \lambda \cdot d \in P} \geq b \ \forall \lambda \geq 0\}$

$= \{d \in \mathbb{R}^n \mid A \cdot d \geq 0\}$ polyhedral cone.

The nonzero elements of the recession cone are called the rays of P .

The recession cone of $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$
is $\{d \in \mathbb{R}^n \mid A \cdot d = 0, d \geq 0\}$.

$$\begin{aligned} A \cdot (y + \lambda \cdot d) \\ &= \underbrace{A \cdot y}_{=b} + \lambda \cdot A \cdot d \end{aligned}$$



Extreme rays

Def: (i) A nonzero element x of a polyhedral cone C is an extreme ray, if there are $n-1$ linearly indep. constraints that are active at x .

(ii) An extreme ray of the recession cone of some polyhedron P is also called an extreme ray of P .

Remark: Up to multiplication with positive vectors there are only finitely many extreme rays of a polyhedron.

Characterization of unbounded linear programs

Theorem: Let $C := \{x \in \mathbb{R}^n \mid a_i^T x \geq 0, i=1, \dots, m\}$ a pointed polyhedral cone and $c \in \mathbb{R}^n$. The minimal cost $c^T x$ s.t. $x \in C$ is equal to $-\infty \iff$ There exists an extreme ray d of C with $c^T d < 0$.

Proof: " \Leftarrow " trivial \checkmark

" \Rightarrow ": $\text{opt} = -\infty \Rightarrow \exists x \in C : c^T x < 0$

$\Rightarrow \exists x \in C : c^T x = -1$

$\Rightarrow P := \{x \in \mathbb{R}^n \mid a_i^T x \geq 0, i=1, \dots, m, c^T x = -1\} \neq \emptyset$

P has at least one extreme point since there are n linearly independent rows in the linear system defining the polyhedron. Let $d \in P$ be an extreme point. \Rightarrow There are n linearly independent active constraints. \Rightarrow Among the constraints $a_i^T x \geq 0$ $i=1, \dots, m$ there are $n-1$ that are active.

$\Rightarrow d$ is extreme ray

\square

Theorem: The last theorem also holds if C is a polyhedron with at least one extreme point.

Proof: Let $C = \{x \in \mathbb{R}^n \mid Ax \geq b\}$

" \Leftarrow " clear.

" \Rightarrow ": Consider infeasible dual LP:

$$\begin{aligned} \max \quad & p^T b \\ \text{s.t.} \quad & p^T A = c^T \\ & p \geq 0 \end{aligned}$$

Changing objective function, the LP remains infeasible:

$$\begin{aligned} \max \quad & p^T 0 \\ \text{s.t.} \quad & p^T A = c^T \\ & p \geq 0 \end{aligned}$$

\rightarrow associated primal LP is infeasible or unbounded

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq 0 \end{aligned}$$

not infeasible since $x=0$ is feasible solution

\rightarrow unbounded.

\rightarrow By the last theorem, there exist an extreme ray d of $\{x \mid Ax \geq 0\}$ with $c^T d < 0$.

d is by def. also an extreme ray of P . \square

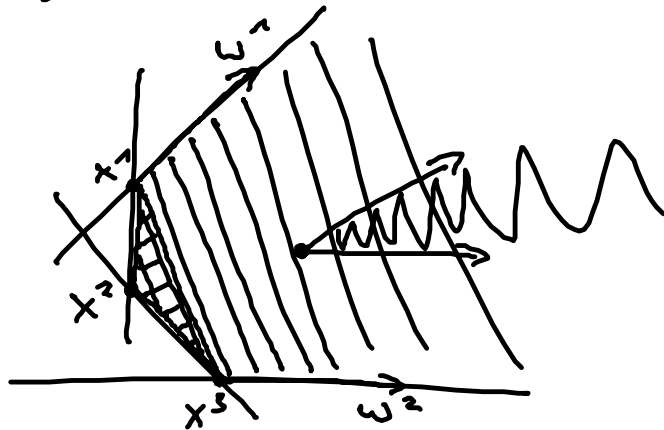
Remark: If the simplex method observes that an LP is unbounded, the corresponding j -th basic direction is an extreme ray d with $c^T d < 0$.

4.9 Representation of polyhedra

Theorem: Let $P = \{x \in \mathbb{R}^n \mid Ax \geq b\} \neq \emptyset$ with at least

one extreme point. Let x^1, \dots, x^k be the extreme points of P and w^1, \dots, w^r a complete set of extreme rays of P . Then:

$$P = \left\{ \sum_{i=1}^k \lambda_i x^i + \sum_{j=1}^r \theta_j w^j \mid \lambda_i, \theta_j \geq 0, \sum_{i=1}^k \lambda_i = 1 \right\} =: Q$$



Proof: " \supseteq " clear.

" \subseteq ": Assume by contradiction that $z \in P \setminus Q$. Consider following LP:

$$\begin{aligned} \max & 0 \\ \text{s.t.} & \sum_{i=1}^k \lambda_i x^i + \sum_{j=1}^r \theta_j w^j = z \\ & \sum_{i=1}^k \lambda_i = 1 \\ & \lambda, \theta \geq 0 \end{aligned}$$

Infeasible since $z \notin Q$. Consider dual LP

$$\begin{aligned} \min & p^T \cdot z + q \cdot 1 \\ \text{s.t.} & p^T \cdot x^i + q \geq 0 \\ & p^T \cdot w^j \geq 0 \end{aligned} \quad \text{feasible } (p=0, q=0)$$

\Rightarrow unbounded $\Rightarrow \exists$ feasible solution (p, q)
with $p^T \cdot z + q < 0$

$$\begin{aligned} p^T \cdot x^i + q &\geq 0 \\ p^T \cdot w^i &\geq 0 \end{aligned}$$

Consider LP:

$$\begin{aligned} \min \quad & p^T \cdot x \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$

z is a feasible sol.
(since $z \in P$)

If LP is bounded \Rightarrow opt. sol. x for some i \Leftarrow
 $p^T z \leq p^T x^i \quad \forall i$

\Rightarrow LP is unbounded $\Rightarrow \exists$ extreme ray w^i with
 $p^T \cdot w^i < 0$ \Leftarrow \square

Corollary: A nonempty bounded polyhedron is the convex hull of its extreme points.

Corollary: If $C = \{x \mid Ax \geq 0\}$ pointed, then every $x \in C$ is a nonnegative linear combination of extreme rays of C .

Theorem: (without proof — see book)

For $x^1, \dots, x^k \in \mathbb{R}^n$, $w^1, \dots, w^r \in \mathbb{R}^n$ the set

$$Q := \left\{ \sum_{i=1}^k \lambda_i x^i + \sum_{j=1}^r \theta_j w^j \mid \lambda_i, \theta_j \geq 0, \sum_{i=1}^k \lambda_i = 1 \right\}$$

is a polyhedron.