

## S.2 Global dependence on the right-hand side

$$\begin{aligned} \min c^T x \\ \text{s.t. } Ax = b \\ x \geq 0 \end{aligned}$$

$$P(b) = \{x \mid Ax = b, x \geq 0\}$$

$$S = \{b \mid P(b) \neq \emptyset\} \quad \text{convex}$$

$$F(b) := \min_{x \in P(b)} c^T x \quad \forall b \in S$$

Assume that dual feasible set is nonempty

$$\{p \mid p^T A \leq c^T\} \neq \emptyset$$

$$\Rightarrow F(b) > -\infty \quad \forall b \in S$$

$b^* \in S$  with non-degenerate optimal basis  $B$

Then  $F$  is linear in the vicinity of  $b^*$  and the gradient of  $F$  in  $b^*$  is equal to the dual solution  $p$ .

Theorem:  $F(b)$  is a convex function of  $b$ .  
and piecewise linear

Proof: Consider the feasible dual LP

$$\max p^T b$$

$$\text{s.t. } p^T A \leq c^T$$

$$\text{rank}(A) = m$$

$\Rightarrow$  extreme points do exist.

Let  $p^1, \dots, p^N$  denote the extreme points of  $\{p \mid p^T A \leq c^T\}$

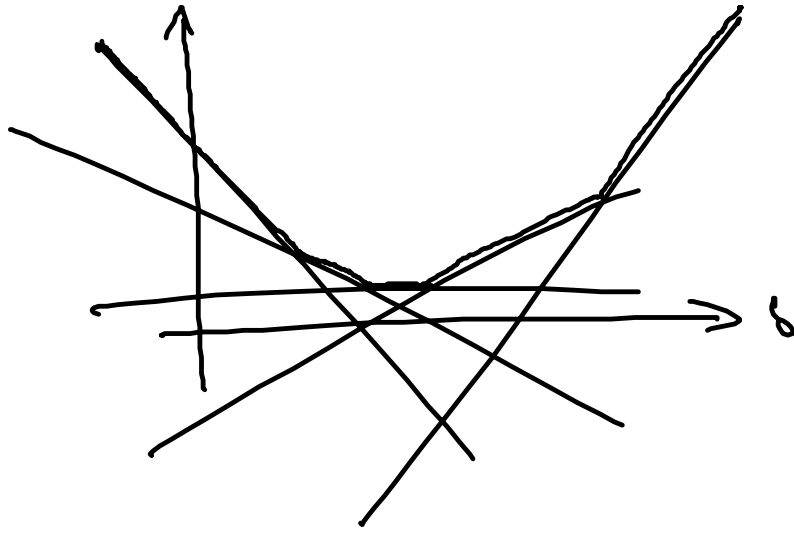
By strong duality

$$F(b) = \max_{i=1, \dots, N} (p^i)^T \cdot b \quad \forall b \in S$$

$\Rightarrow F$  is maximum of convex (linear) functions

$\Rightarrow F$  is convex.  
 $F(b)$

□

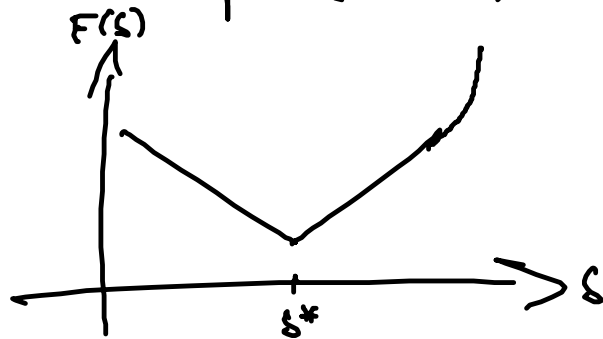


### 5.3 The set of all dual optimal solutions

$F$  linear in the vicinity of  $s \Rightarrow$  opt. dual solution  $p$  is gradient of  $F$  in point  $s$ .

Def: Let  $S \subseteq \mathbb{R}^n$  convex and  $F: S \rightarrow \mathbb{R}$  convex and  $s^* \in S$ . Then  $p$  is a subgradient of  $F$  at  $s^*$  if

$$F(s^*) + p^T \cdot (s - s^*) \leq F(s) \quad \forall s \in S$$



Remark: If  $F$  is linear (or at least differentiable) in  $s^*$  then there is a unique subgradient, namely the gradient.

If  $s^*$  is a breakpoint of  $F$ , then there are several subgradients.

Theorem: Suppose that the LP  $\min c^T x$  s.t.  $Ax = b^*$   
 $x \geq 0$   
 is feasible and bounded. Then  $p$  is an optimal solution to the dual LP if and only if  $p$  is a subgradient of  $F$  at  $b^*$ .

Proof:  $\Rightarrow$ : Let  $p$  be an opt. dual solution

$$\Rightarrow p^T \cdot b^* = F(b^*). \text{ For } b \in S \text{ and } x \in P(b)$$

$$p^T \cdot b \leq c^T x \quad (\text{weak duality})$$

$$\Rightarrow p^T \cdot b \leq F(b) = \min_{x \in P(b)} c^T x$$

$$\Rightarrow F(b^*) - p^T b^* + p^T b \leq F(b)$$

$\Leftarrow$ : Assume that  $p$  subgradient of  $F$  at  $b^*$

$$F(b^*) + p^T (b - b^*) \leq F(b) \quad \forall b \in S.$$

For some  $x \geq 0$  let  $b := A \cdot x \Rightarrow x \in P(b)$

$$\Rightarrow F(b) \leq c^T x$$

$$\Rightarrow p^T \cdot A \cdot x = p^T \cdot b \leq F(b) - F(b^*) + p^T \cdot b^*$$

$$\leq c^T x - F(b^*) + p^T \cdot b^*$$

$$\Rightarrow p^T A x \leq c^T x - F(b^*) + p^T b^* \quad \forall x \geq 0$$

$$\Rightarrow p^T A \leq c^T \quad (\text{i.e. } p \text{ is feasible dual sol.})$$

For  $x=0$  we get  $0 \leq -F(b^*) + p^T \cdot b^*$

$$\Rightarrow p^T \cdot b^* \geq F(b^*) \Rightarrow p \text{ optimal. } \square$$

## 5.4 Global dependence on the cost vector

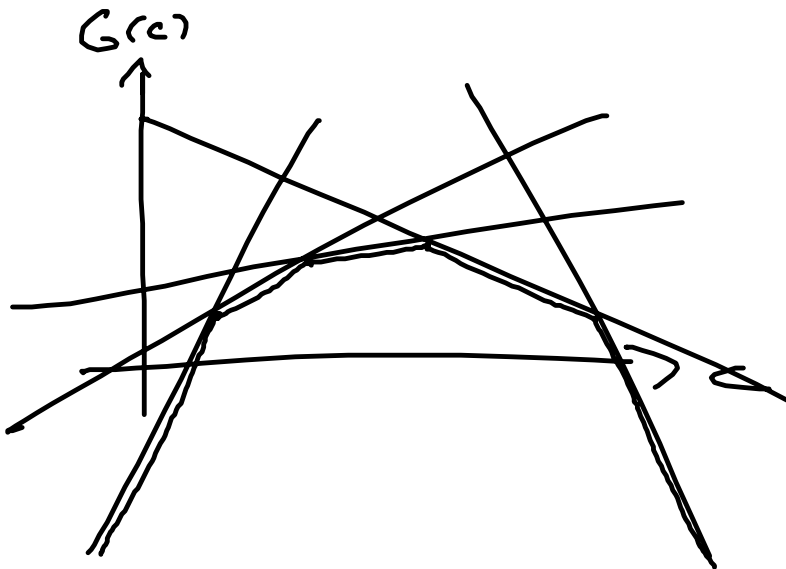
Let  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$  such that  $\{x \mid Ax=b, x \geq 0\} \neq \emptyset$   
 and for  $c \in \mathbb{R}^n$

$$Q(c) := \{p \mid p^T A \leq c^T\}$$

$$T := \{c \mid Q(c) \neq \emptyset\} \text{ convex}$$

Note that  $T := \{c \mid \underbrace{\min c^T x \text{ s.t. } Ax=b, x \geq 0 \text{ is finite}}_{=: G(c)}\}$   
 $G(c) = \min_{i=1, \dots, N} c^T x^i$

where  $x^1, \dots, x^N$  are the basic feasible solutions.



Theorem: For a feasible LP in standard form

- (i)  $T$  is convex
- (ii)  $G(c)$  is a piecewise linear concave function of  $c$ .
- (iii) If for some  $c \in T$  the primal LP has a unique opt. sol.  $x^*$ , then  $G$  is linear in the vicinity of  $c$  and its gradient is equal to  $x^*$ .

# S.S. Parametric programming

Let  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$  and  $c, d \in \mathbb{R}^n$

Task: Solve

$$g(\theta) := \min (c + \theta \cdot d)^T \cdot x$$
$$\text{s.t. } Ax = b$$
$$x \geq 0$$

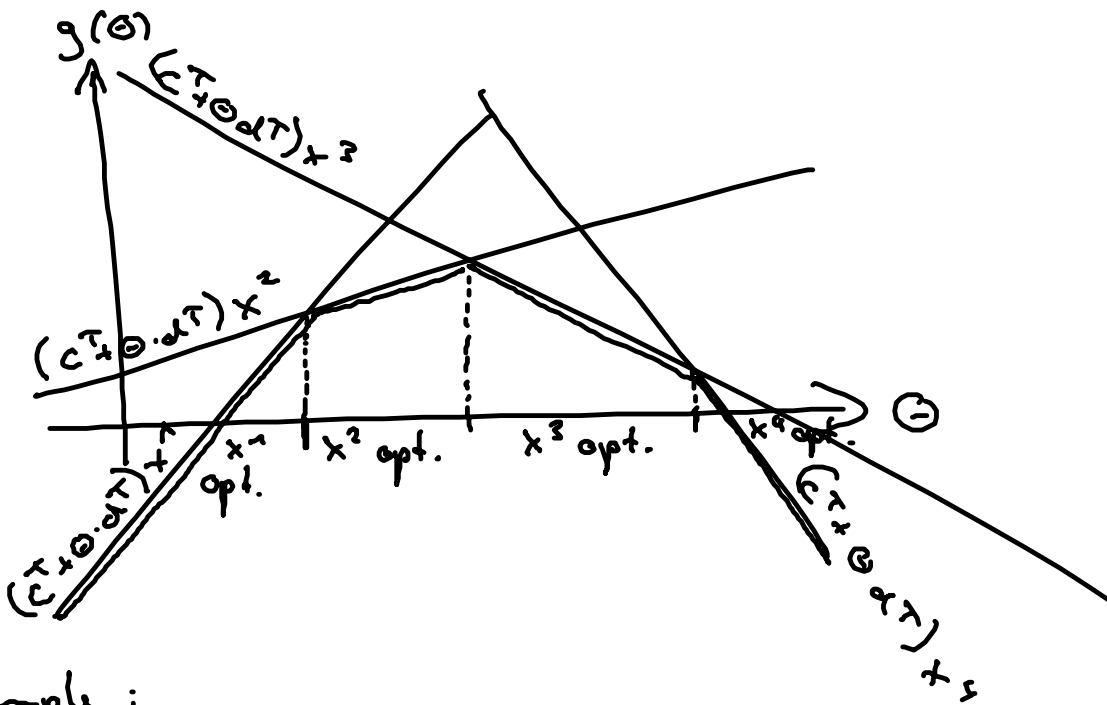
for all  $\theta \in \mathbb{R}$  with  $g(\theta) > -\infty$ .

Assume that  $\{x \mid Ax = b, x \geq 0\} \neq \emptyset$ .

Notice that

$$g(\theta) = \min_{i=1, \dots, N} (c^T + \theta d^T) \cdot x^i$$

where  $x^1, \dots, x^N$  are basic feas. sol.



Example:

$$\min (-3 + 2\theta)x_1 + (3 - \theta)x_2 + x_3$$

s.t.

$$x_1 + 2x_2 - 3x_3 \leq 5$$
$$2x_1 + x_2 - 4x_3 \leq 7$$

$$x_1, x_2, x_3 \geq 0$$

→ Tableau

	$\theta$	$-3+2\theta$	$3-\theta$	1	0	0
$x_4 =$	5	1	2	-3	1	0
$x_5 =$	7	2	1	-4	0	1

If  $-3+2\theta \geq 0$  and  $3-\theta \geq 0$ , then this sol. is optimal

$$\Rightarrow g(\theta) = 0 \text{ for } \theta \in \left[\frac{3}{2}, 3\right]$$

Consider  $\theta > 3 \Rightarrow \bar{c}_2 < 0$  do pivoting step

→

	$-7.5+2.5\theta$	$-4.5+2.5\theta$	0	$5.5-1.5\theta$	$-1.5+0.5\theta$	0
$x_2 =$	2.5	0.5	1	-1.5	0.5	0
$x_5 =$	4.5	1.5	0	-2.5	-0.5	1

optimal for  $3 \leq \theta \leq \frac{5.5}{1.5} = \frac{11}{3}$

$$\Rightarrow g(\theta) = 7.5 - 2.5 \cdot \theta \text{ for } \theta \in \left[3, \frac{11}{3}\right]$$

Consider  $\theta > \frac{11}{3} \Rightarrow \bar{c}_3 < 0$  and the LP is unbounded

$$g(\theta) = -\infty \text{ for } \theta \in \left(\frac{11}{3}, \infty\right)$$

Go back to initial tableau and consider

$\theta < \frac{3}{2} \Rightarrow \bar{c}_1 < 0$  do pivoting operation

→

$10.5 - 7 \cdot \theta$	0	$4.5 - 2\theta$	$-5 + 4\theta$	0	$1.5 - \theta$
$x_4 = 1.5$	0	1.5	-1	1	-0.5
$x_1 = 3.5$	1	0.5	-2	0	0.5

optimal for  $\frac{5}{4} \leq \theta \leq \frac{3}{2}$

$$g(\theta) = -10.5 + 7 \cdot \theta \quad \text{for } \theta \in \left[ \frac{5}{4}, \frac{3}{2} \right]$$

Consider  $\theta < \frac{5}{4} \Rightarrow \bar{c}_3 < 0$  and the LP is unbounded

$$g(\theta) = -\infty \quad \text{for } \theta \in (-\infty, \frac{5}{4}).$$

This approach works in general and terminates after a finite number of steps since no basis is visited more than once.

(avoid cycling by using, e.g., lexicographic rule.)