

Chapter 8: Complexity of linear progr. and the ellipsoid method

The Klee-Minty cubes show that the simplex has a quite sad worst-case behavior. In this chapter we discuss an efficient algorithm for linear prog.

8.1. Running time of algorithms

What is an "efficient" "algorithm"

↑
running times

↑
Turing machines
or other formal
model of computation

For us, an algorithm A consists of "elementary" steps, i.e., for example variable assignment and simple arithmetic operations which only take a constant amount of time.

bit model

arithmetic model

count bit operations
(e.g. adding two bits etc.)
adding two u -bit numbers
takes $u+1$ steps; multi-
plying these numbers takes
 $O(u^2)$ steps.

simple arithmetic
operations on
arbitrary numbers
can be performed
in constant time.

The number of elementary steps of A depends
on the input I , in particular on the size
of I , denoted by $\text{size}(I)$.

bit model

$\text{size}(I) = \# \text{ bits}$
(in a predefined
format)

arithmetic model

$\text{size}(I) = \# \text{ bits to encode}$
structure of $I +$
 $\# \text{ numbers.}$

The running time of A is bounded by
 $f: \mathbb{N} \rightarrow \mathbb{N}$ if A performs at most $f(u)$
elementary steps on any input of size $\leq u$.
"worst case running time"

Def ("Efficient algorithm")

An algorithm runs in polynomial time, if its running time is bounded by a polynomial in the bit model. It even runs in strongly polynomial time, if its running time is bounded by a polynomial both in the bit model and the arithmetic model.

Remark: An algorithm that is polynomial in the arithmetic model and has the property that the size of any number produced in the course of this algorithm is bounded by a polynomial in the input size, runs in strongly polynomial time.

8.2 The key geometric result behind the ellipsoid method

Def: $D \in \mathbb{R}^{n \times n}$ symmetric is positive definite if $x^T \cdot D \cdot x > 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}$

Remark: The following are equivalent for a symmetric $D \in \mathbb{R}^{n \times n}$

(i) D is positive definite

(ii) D^{-1} is positive definite

(iii) D has only real and positive eigenvalues.

(iv) $D = B^T \cdot B$ for a nonsingular matrix $B \in \mathbb{R}^{n \times n}$.

Def: A set $E \subseteq \mathbb{R}^n$ of the form

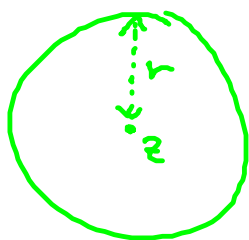
$$E = E(z, D) := \{x \in \mathbb{R}^n \mid (x-z)^T \cdot D^{-1} \cdot (x-z) \leq 1\}$$

where $D \in \mathbb{R}^{n \times n}$ is positive definite is called an ellipsoid with center z .

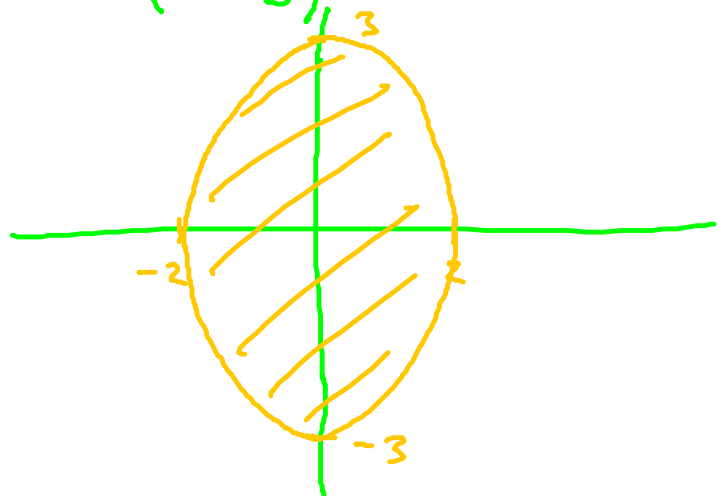
Example: (i) For $r > 0$, the ellipsoid

$$\begin{aligned} E(z, r^2 \cdot I) &= \{x \in \mathbb{R}^n \mid (x-z)^T (x-z) \leq r^2\} \\ &= \{x \in \mathbb{R}^n \mid \|x-z\|_2 \leq r\} \end{aligned}$$

is the ball of radius r centered at z .



(ii) $D = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$ $z = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



Def: If $D \in \mathbb{R}^{n \times n}$ is nonsingular and $b \in \mathbb{R}^n$
then the mapping $S: \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined
by $S(x) := Dx + b$ is an affine transform.

For $L \subseteq \mathbb{R}^n$ let $S(L) := \{y \in \mathbb{R}^n \mid y = Dx + b$
for some
 $x \in \mathbb{R}^n\}$

Define the volume of $L \subseteq \mathbb{R}^n$ as

$$\text{Vol}(L) = \int_{x \in L} 1 \cdot dx$$

Lemma: If $S(x) = D \cdot x + b$ then

$$\text{Vol}(S(L)) = |\det D| \cdot \text{Vol}(L).$$

Proof: $\text{Vol}(S(L)) = \int_{y \in S(L)} 1 \cdot dy$

$$= \int_{x \in L} |\det(J(x))| dx$$

where $J(x)$ is the Jacobian matrix associated with S , i.e.,

$$(J(x))_{ij} = \frac{\partial S_i(x)}{\partial x_j} = D_{ij}$$

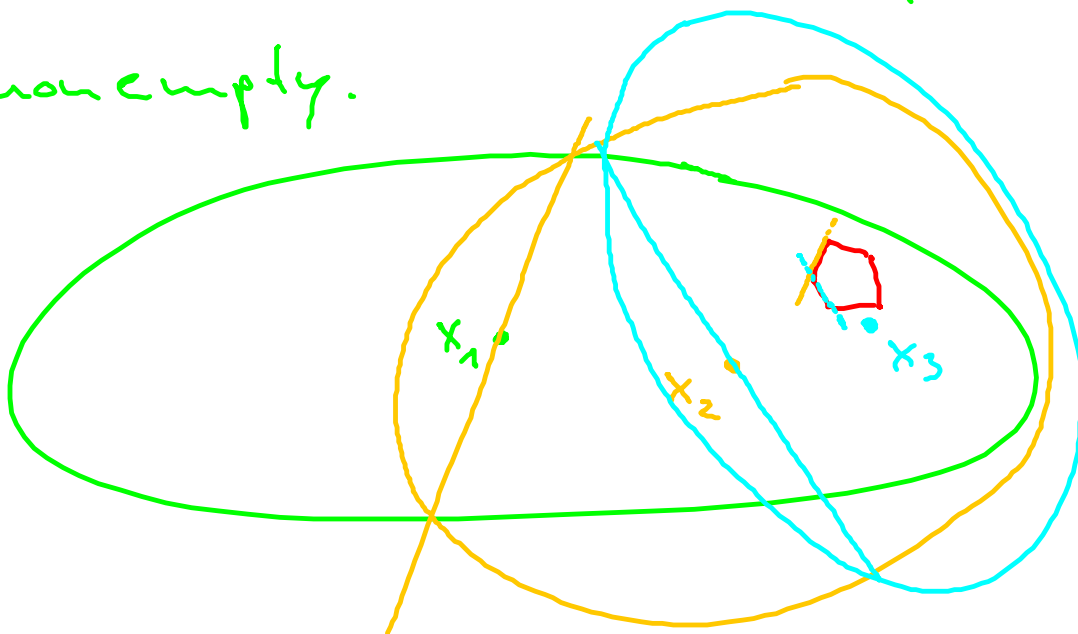
$$= \int_{x \in L} |\det D| dx = |\det D| \text{vol}(L). \quad \square$$

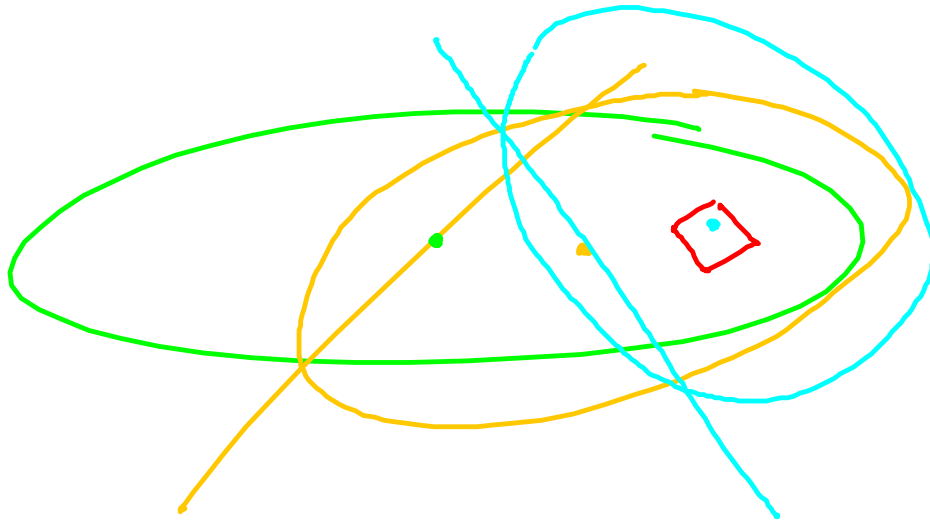
Rough idea of the ellipsoid method:

The ellipsoid method can be used to decide whether a given polyhedron

$$P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$$

is nonempty.





The key insight for the ellipsoid method is that the volume of the ellipsoids strictly decreases in every iteration.