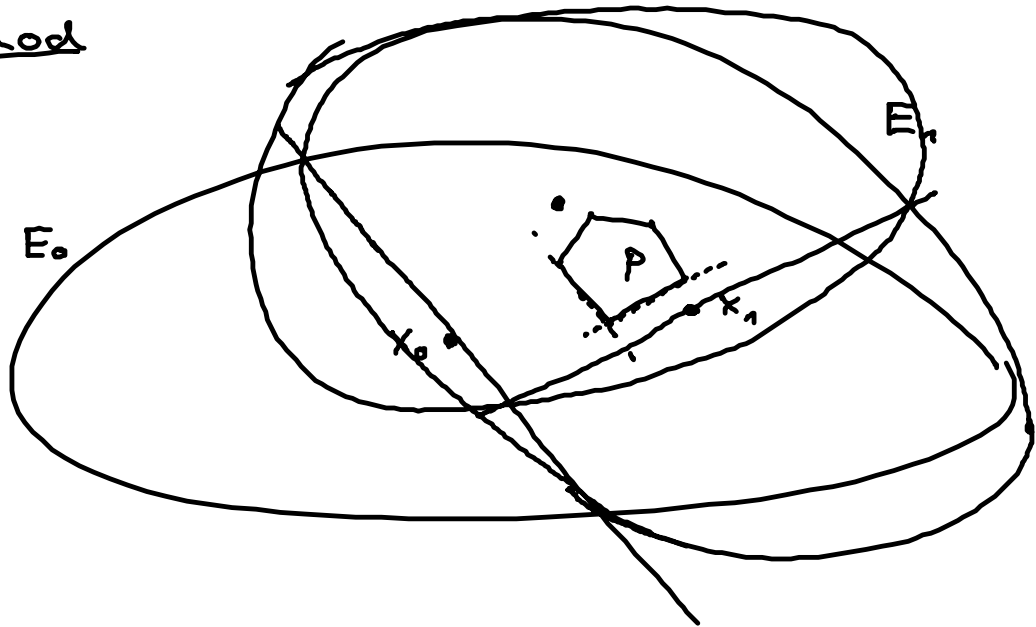


# Ellipsoid method



## Theorem:

Let  $E = E(z, D)$  be an ellipsoid in  $\mathbb{R}^n$  and  $a \in \mathbb{R}^n \setminus \{0\}$ .

Let  $H := \{x \in \mathbb{R}^n \mid a^T x \geq a^T z\}$  and

$$\bar{z} := z + \frac{1}{n+1} \cdot \frac{D \cdot a}{\sqrt{a^T \cdot D \cdot a}}$$

$$\bar{D} := \frac{n^2}{n^2-1} \cdot \left( D - \frac{2}{n+1} \cdot \frac{D a \cdot a^T \cdot D}{a^T \cdot D \cdot a} \right)$$

The matrix  $\bar{D}$  is symmetric and positive definite and thus  $\bar{E} = (\bar{z}, \bar{D})$  is an ellipsoid. Moreover

(i)  $E \cap H \subseteq \bar{E}$

(ii)  $\text{Vol}(\bar{E}) < e^{-\frac{1}{2(n+1)}} \cdot \text{Vol}(E)$

Proof: see book.

## 8.3 The ellipsoid method for the feasibility problem

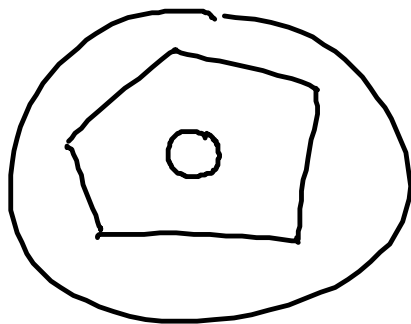
□

Let  $P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$

Def: A polyhedron  $P$  is full-dimensional if it has positive volume.

## Simplifying Assumptions

(i)  $P$  is bounded and either empty or full-dimensional.



In particular  $P \subseteq E(x_0, r^2 I) =: E_0$  for some  $r > 0$ ,  $\text{Vol}(E_0) \leq V$  for some  $V \in \mathbb{R}$ ;

$P$  either empty or  $\text{Vol}(P) \geq v$  for some  $v > 0$ .

Assume that  $E_0, V, v$  are known a priori.

(ii) Calculations (including square roots) can be made in infinite precision.

### Ellipsoid method

- 1)  $t^* := \lceil 2(n+1) \cdot \log \frac{V}{v} \rceil$ ;  $E_0 := E(x_0, r^2 \cdot I)$ ,  $D_0 := r^2 \cdot I$ ;  $t=0$
- 2) a) If  $t = t^*$  stop;  $P$  is empty.  
b) If  $x_t \in P$  stop;  $P$  is non-empty.  
c) If  $x_t \notin P$  then find violated constraint  $a_i^T \cdot x_t < b_i$ ;  
d)  $H_t := \{x \in \mathbb{R}^n \mid a_i^T x \geq a_i^T x_t\}$ ; find ellipsoid  $E_{t+1}$  containing  $E_t \cap H_t$  by applying the theorem above.  
e)  $t := t+1$ ; goto a).

Theorem: The ellipsoid method decides correctly whether  $P$  is empty or not.

Proof: If  $x_t \in P$  for some  $t < t^*$ , then the answer given by the algorithm is correct. ✓  
Otherwise:

By induction  $P \subseteq E_k$  for  $k=0,1,\dots,t^*$

Since  $\text{Vol}(E_{k+1})/\text{Vol}(E_k) < e^{-\frac{1}{2(u+1)}}$  by Th. 8.1

$$\frac{\text{Vol}(E_{t^*})}{\text{Vol}(E_0)} < e^{-\frac{t^*}{2(u+1)}}$$

$$\begin{aligned} \Rightarrow \text{Vol}(E_{t^*}) &< V \cdot e^{-\lceil 2(u+1) \log V/v \rceil / (2(u+1))} \\ &\leq V \cdot e^{-\log V/v} = v \end{aligned}$$

If  $E_{t^*}$  contains  $P$ , then  $P$  is empty.  $\square$

Why can we assume that  $P$  is bounded?

Lemma:  $A \in \mathbb{Z}^{u \times u}$ ,  $b \in \mathbb{R}^u$

$U :=$  largest absolute value of an entry of  $A$  and  $b$ .

(i) Every extreme point of  $P = \{x \mid Ax \geq b\}$  satisfies

$$-(u \cdot U)^u \leq x_j \leq (u \cdot U)^u, j=1, \dots, u$$

$$(U := \max \{|a_{ij}|, |b_i| \mid i=1, \dots, u, j=1, \dots, u\})$$

(ii) Every extreme point of  $P' = \{x \mid Ax = b, x \geq 0\}$  satisfies:

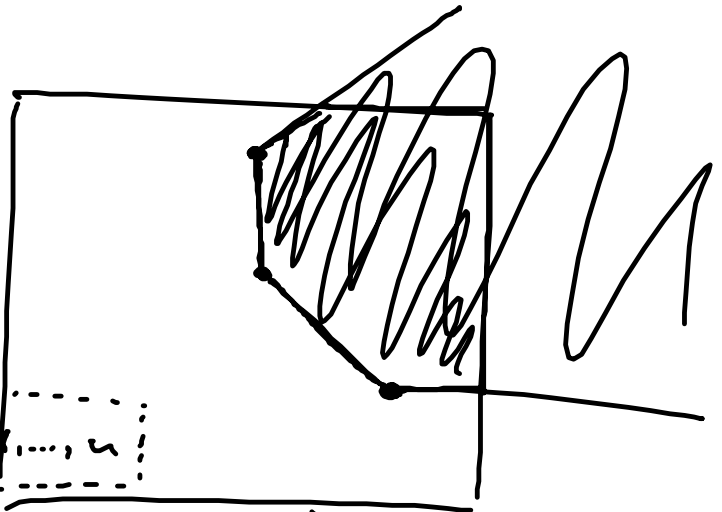
$$-(u \cdot U)^u \leq x_j \leq (u \cdot U)^u \text{ for } j=1, \dots, u$$

Proof: (i) There exist  $u$  linearly independent active inequalities.

$$\text{at } x : \hat{A} \cdot x = \hat{b} \quad \rightarrow \quad x = \hat{A}^{-1} \cdot \hat{b}$$

$$\text{Cramer's rule: } x_j = \frac{\det(\hat{A}^j)}{\det(\hat{A})} \quad (*)$$

$$\det(\hat{A}^j) = \sum_{\sigma \in S_u} \text{sgn}(\sigma) \cdot \prod_{i=1}^u \hat{a}_{i, \sigma(i)}$$



$$\Rightarrow |\det(\hat{A}^j)| \leq \sum_{G \in S_n} \prod_{i=1}^n |\hat{\alpha}_{i, G(i)}| \leq n! \cdot U^n \leq (n \cdot U)^n$$

Since  $\hat{A} \in \mathbb{Z}^{n \times n}$  nonsingular,  $|\det \hat{A}| \geq 1$

$$\stackrel{(*)}{\Rightarrow} |x_j| \leq (n \cdot U)^n$$

(ii) similar to (i)  $\square$

Define  $P_B := \{x \in P \mid |x_j| \leq (n \cdot U)^n \text{ for } j=1, \dots, n\}$   
 bounded polyhedron. Under the assumption that  
 the rows of  $A$  span  $\mathbb{R}^n$ :

$P \neq \emptyset \Leftrightarrow P$  contains extreme point

$$\Leftrightarrow P_B \neq \emptyset$$

$\rightarrow$  work with  $P_B$  instead of  $P$ .

Start with ellipsoid  $E_0 = E(0, n \cdot (n \cdot U)^{2n} \cdot \mathbb{I}) \supseteq P_B$ .

Note that  $\text{Vol}(E_0) \leq V := (2n(n \cdot U)^n)^n = (2n)^n \cdot (n \cdot U)^{n^2}$ .

Why can we assume that  $P$  is full-dimensional or empty?

Lemma:  $A \in \mathbb{Z}^{m \times n}$ ,  $b \in \mathbb{Z}^m$ ,  $U :=$  (see above)

Let  $P = \{x \mid Ax \geq b\}$

$$\varepsilon := \frac{1}{2^{(n+1)}} \cdot (n+1) \cdot U^{-(n+1)}$$

and

$$P_\varepsilon = \{x \mid Ax \geq b - \varepsilon \cdot \mathbb{1}\}$$

(i)  $P$  empty  $\Rightarrow P_\varepsilon$  empty

(ii)  $P \neq \emptyset \Rightarrow P_\varepsilon$  full-dimensional.