Ellipsoid method

Theorem:

Let $E = E(z, D)$ be an ellipsoid in $\mathbb{R}^n$ and $a \in \mathbb{R}^n \setminus \{0\}$.
Let $H := \{x \in \mathbb{R}^n \mid a^T x \geq a^T z\}$ and

$$\bar{z} := z + \frac{1}{n+1} \cdot \frac{D \cdot a}{\sqrt{a^T \cdot D \cdot a}}$$

$$\bar{D} := \frac{n^2}{n^2-1} \cdot \left(D - \frac{2}{n+1} \cdot \frac{D a \cdot a^T \cdot D}{a^T \cdot D \cdot a} \right)$$

The matrix $\bar{D}$ is symmetric and positive definite and thus $\bar{E} = (\bar{z}, \bar{D})$ is an ellipsoid. Moreover

(i) $E \cap H \subseteq \bar{E}$

(ii) $\mathrm{Vol}(\bar{E}) < e^{-\frac{1}{2(n+1)}} \cdot \mathrm{Vol}(E)$

Proof: see book. □

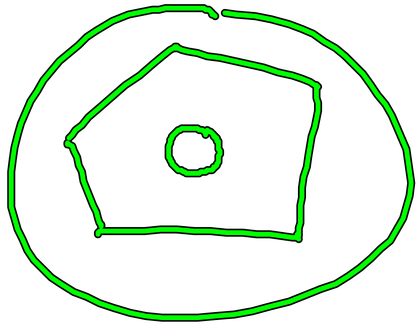
8.3 The ellipsoid method for the feasibility problem

Let $P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$

Def: A polyhedron P is full-dimensional if it has positive volume.

Simplifying Assumptions

(i) P is bounded and either empty or full-dimensional.



In particular $P \subseteq E(x_0, r^2 I) =: E_0$ for some $r > 0$, $Vol(E_0) \leq V$ for some $V \in \mathbb{R}$;
P either empty or $Vol(P) \geq v$ for some $v > 0$.

Assume that E_0, V, v are known a priori.

(ii) Calculations (including square roots) can be made in infinite precision.

Ellipsoid method

1) $t^* := \lceil 2(n+1) \cdot \log \frac{V}{v} \rceil$; $E_0 := E(x_0, r^2 \cdot I)$, $D_0 := r^2 \cdot I$; $t = 0$

2) a) If $t = t^*$ stop; P is empty.
   b) If $x_t \in P$ stop; P is non-empty.
   c) If $x_t \notin P$ then find violated constraint $a_i^T \cdot x_t < b_i$
   d) $H_t := \{x \in \mathbb{R}^n \mid a_i^T x \geq a_i^T \cdot x_t\}$; find ellipsoid E_{t+1} containing $E_t \cap H_t$ by applying the theorem above.
   e) $t := t+1$; goto a).

Theorem: The ellipsoid method decides correctly whether P is empty or not.

Proof: If $x_t \in P$ for some $t < t^*$, then the answer given by the algorithm is correct. ✓
Otherwise:

By induction $P \subseteq E_k$ for $k = 0, 1, \dots, t^*$

Since $\mathrm{Vol}(E_{k+1}) / \mathrm{Vol}(E_k) < e^{-\frac{1}{2(n+1)}}$ by Th. 8.1

$$\frac{\mathrm{Vol}(E_{t^*})}{\mathrm{Vol}(E_0)} < e^{-\frac{t^*}{2(n+1)}}$$

$$\Rightarrow \mathrm{Vol}(E_{t^*}) < V \cdot e^{-\lceil 2(n+1) \log V/v \rceil / (2(n+1))}$$
$$\leq V \cdot e^{-\log V/v} = v$$

If E_{t^*} contains P, then P is empty. $\square$

Why can we assume that P is bounded?

Lemma: $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{R}^m$

$U :=$ largest absolute value of an entry of A and b.

(i) Every extreme point of $P = \{x \mid Ax \geq b\}$ satisfies

$$\boxed{-(n \cdot U)^n \leq x_j \leq (n \cdot U)^n, \; j = 1, \dots, n}$$

$\left(U := \max\{|a_{ij}|, |b_i| \mid i = 1, \dots, m, \; j = 1, \dots, n\} \right)$

(ii) Every extreme point of $P' = \{x \mid Ax = b, x \geq 0\}$ satisfies:

$$\boxed{-(m \cdot U)^m \leq x_j \leq (m \cdot U)^m \quad \text{for } j = 1, \dots, n}$$

Proof: (i) There exist n linearly independent active inequal. at x: $\hat{A} \cdot x = \hat{b} \Rightarrow x = \hat{A}^{-1} \cdot \hat{b}$

Cramer's rule: $$x_j = \frac{\det(\hat{A}^j)}{\det(\hat{A})} \quad (*)$$

$$\det(\hat{A}^j) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot \prod_{i=1}^{n} \tilde{a}_{i,\sigma(i)}$$

$$\Rightarrow |\det(\hat{A}^j)| \leq \sum_{\sigma \in S_n} \prod_{i=1}^{n} \underbrace{|\tilde{a}_{i,\sigma(i)}|}_{\leq U} \leq n! \cdot U^n \leq (n \cdot U)^n$$

Since $\hat{A} \in \mathbb{Z}^{n \times n}$ nonsingular, $|\det \hat{A}| \geq 1$

$$\overset{(*)}{\Longrightarrow} |x_j| \leq (n \cdot U)^n$$

(ii) similar to (i) $\square$

Define $P_B := \{x \in P \mid |x_j| \leq (n \cdot U)^n \text{ for } j = 1, \dots, n\}$

bounded polyhedron. Under the assumption that the rows of A span $\mathbb{R}^n$:

$$P \neq \emptyset \Longleftrightarrow P \text{ contains extreme point}$$
$$\Longleftrightarrow P_B \neq \emptyset$$

$\longrightarrow$ work with P_B instead of P.

Start with ellipsoid $E_0 = E(0, n \cdot (n \cdot U)^{2n} \cdot I) \supseteq P_B$.

Note that $\mathrm{Vol}(E_0) \leq V := (2n(nU)^n)^n = (2n)^n \cdot (n \cdot U)^{n^2}$.

<u>Why can we assume that P is full-dimensional or empty</u> ?

<u>Lemma:</u> $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $U :=$ (see above)

Let $P = \{x \mid Ax \geq b\}$

$$\varepsilon := \frac{1}{2(n+1)} \cdot ((n+1) \cdot U)^{-(n+1)}$$

and

$$P_\varepsilon = \{x \mid Ax \geq b - \varepsilon \cdot \mathbb{1}\}$$

(i) P empty $\Rightarrow P_\varepsilon$ empty

(ii) $P \neq \emptyset \Rightarrow P_\varepsilon$ full-dimensional.