

# Chapter 7. Network Simplex

## 7.1. Basic definitions

Undirected Graphs.  $G=(V,E)$  is an undirected graph if

$E \subseteq \binom{V}{2} = \{ \{u,v\} \mid u,v \in V, u \neq v \}$ . Nodes  $u,v$  are incident to the edge  $\{u,v\}$ .  $S(v) = \{ \{v,u\} \in E \}$ , the degree of  $v$  is  $\deg(v) = |S(v)|$ .

A path is a sequence  $v_1, \dots, v_k$  of nodes such that  $\{v_i, v_{i+1}\} \in E$   $\forall i$  and  $v_i \neq v_j$  for  $i \neq j$ .

A cycle is a sequence  $v_1, \dots, v_k$  of nodes such that  $v_1, \dots, v_{k-1}$  form a path and  $v_k = v_1$  and  $\{v_{k-1}, v_1\} \in E$  and  $k \geq 4$ .

An undirected graph is connected if there exists a path from every node to every other.

Directed Graphs.  $G=(V,E)$  is a directed graph if

$E \subseteq V \times V = \{ (u,v) \mid u,v \in V, u \neq v \}$ . For an edge  $(u,v)$  the node  $u$  is called the start node and  $v$  is the end node, and  $(u,v)$  is outgoing from  $u$  and incoming into  $v$ .

$S^+(v) = \{ (v,u) \in E \}$ ,  $S^-(v) = \{ (u,v) \in E \}$ .

For a directed graph construct the underlying undirected graph by ignoring the directions of the edges and delete possible "double edges".

A directed graph is connected if the underlying undirected graph is connected.

A  $\left\{ \begin{array}{l} \text{path} \\ \text{cycle} \end{array} \right\}$  in a directed graph is a sequence of nodes that forms

a  $\left\{ \begin{array}{l} \text{path} \\ \text{cycle} \end{array} \right\}$  in the underlying undirected graph plus a sequence of connecting arcs between these nodes.

Exception: cycles of length 2 are allowed!

A path (or cycle) can have forward and backward edges.

A path with only forward arcs is called directed.

Trees. A tree is an undirected graph that is connected and has no cycles. A node of degree 1 in a tree is called a leaf.

Theorem.

- (a) Every tree with more than one node has at least one leaf.  
(in fact: two leaves!)
- (b) An undirected graph with  $n$  nodes is a tree if and only if it is connected and has  $n-1$  edges.
- (c) Given two nodes  $u, v$  in a tree,  $u \neq v$ , there exists a unique path from  $u$  to  $v$ .
- (d) If an undirected graph  $G$  arises from a tree by adding one new edge, then  $G$  has exactly one cycle.

Let  $G = (V, E)$  be a connected undirected graph. A tree  $T = (V, E')$  with  $E' \subseteq E$  is called a spanning tree.

Theorem. Let  $G = (V, E)$  be a connected undirected graph and  $E_0 \subseteq E$  a set of edges that forms no cycles. Then there exists a set  $E' \subseteq E$ ,  $E_0 \subseteq E'$  such that  $T := (V, E')$  is a spanning tree of  $G$ .

## 7.2 The network flow problem

A network is a directed graph  $G = (V, E)$  together with additional numerical information:

$$\bullet b_v \in \mathbb{R} \quad \forall v \in V \quad (\text{in-flow at node } v)$$

$$\bullet u_e \geq 0 \text{ or } \infty \quad \forall e \in E \quad (\text{capacity of edge } e)$$

$$\bullet c_e \in \mathbb{R} \quad \forall e \in E \quad (\text{cost of edge } e)$$

A node with  $b_v > 0$  is called a source and a sink if  $b_v < 0$ .

In the network flow problem we are searching for a flow  $(f_e)_{e \in E} \in \mathbb{R}^E$ .

A vector  $(f_e)_{e \in E}$  is a feasible flow if these constraints are satisfied:

$$\sum_{e \in \mathcal{S}^+(v)} f_e - \sum_{e \in \mathcal{S}^-(v)} f_e = b_v \quad \forall v \in V \quad (\text{flow conservation})$$

$$f_e \leq u_e \quad \forall e \in E$$

$$f_e \geq 0 \quad \forall e \in E$$

$$\text{Note: } \sum_{v \in V} \left( \sum_{e \in \mathcal{S}^+(v)} f_e - \sum_{e \in \mathcal{S}^-(v)} f_e \right) \stackrel{!}{=} \sum_{e \in E} f_e - f_e = 0$$

|| ← flow conservation

$$\sum_{v \in V} b_v$$

The general minimal cost network flow problem:

$$\min \sum_{e \in E} c_e f_e$$

$$\text{s.t. } \sum_{e \in \mathcal{S}^+(v)} f_e - \sum_{e \in \mathcal{S}^-(v)} f_e = b_v \quad \forall v \in V$$

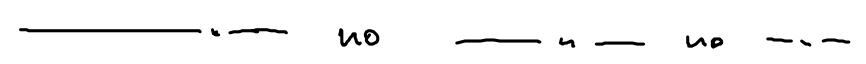
$$\left. \begin{array}{l} f_e \leq u_e \\ f_e \geq 0 \end{array} \right\} \forall e \in E$$

A linear program in standard form if  $u_e = \infty \quad \forall e \in E$ .  
(uncapacitated network)

Special cases:

- shortest path
- max. flow
- transportation problem (equivalent!)
- assignment problem

Variants (equivalent)

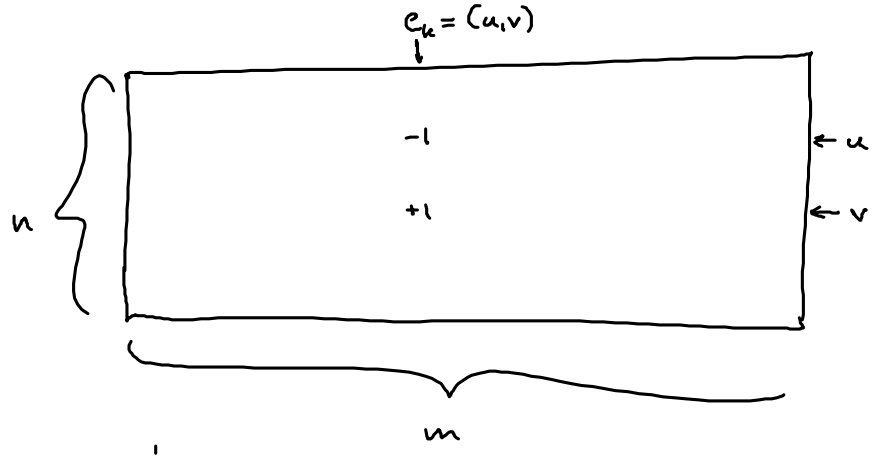
- transportation problem
- problem with only one source and one sink
-  (circulation problem)
- network with node capacities
- network with lower bounds on the edge flows:  $f_e \geq d_e$

Matrix formulation.

W.l.o.g. assume  $V = \{1, \dots, n\}$  and  $|E| = m \geq n-1$ , edges somehow ordered:  $E = \{e_1, \dots, e_m\}$

Incidence matrix:  $A = (a_{i,k})_{\substack{1 \leq i \leq n \\ 1 \leq k \leq m}}$  with

$$a_{i,k} := \begin{cases} +1 & \text{if } e_k = (i, v) \quad \text{for some } v \in V \\ -1 & \text{if } e_k = (v, i) \quad \text{--- } v \text{ ---} \\ 0 & \text{otherwise} \end{cases}$$



If  $f = (f_{e_1}, \dots, f_{e_m})^T$  is a flow vector then flow conservation states

$$A \cdot f = b$$

The rows of  $A$  add up to the zero vector  $\Rightarrow \text{rank } A \leq n - 1$

A flow vector  $f$  with  $A \cdot f = 0$  is called a circulation.