

Chapter 7. Network Simplex

7.1. Basic definitions

Undirected Graphs. $G=(V,E)$ is an undirected graph if $E \subseteq \binom{V}{2} = \{ \{u,v\} \mid u,v \in V, u \neq v \}$. Nodes u,v are incident to the edge $\{u,v\}$. $\mathcal{S}(v) = \{ \{v,u\} \in E \}$, the degree of v is $\deg(v) = |\mathcal{S}(v)|$.

A path is a sequence v_1, \dots, v_k of nodes such that $\{v_i, v_{i+1}\} \in E$ $\forall i$ and $v_i \neq v_j$ for $i \neq j$.

A cycle is a sequence v_1, \dots, v_k of nodes such that v_1, \dots, v_{k-1} form a path and $v_k = v_1$ and $\{v_{k-1}, v_1\} \in E$ and $k \geq 4$. An undirected graph is connected if there exists a path from every node to every other.

Directed Graphs. $G=(V,E)$ is a directed graph if $E \subseteq V \times V = \{ (u,v) \mid u,v \in V, u \neq v \}$. For an edge (u,v) the node u is called the start node and v is the end node, and (u,v) is outgoing from u and incoming into v . $\mathcal{S}^+(v) = \{ (v,u) \in E \}$, $\mathcal{S}^-(v) = \{ (u,v) \in E \}$.

For a directed graph construct the underlying undirected graph by ignoring the directions of the edges and delete possible "double edges".

A directed graph is connected if the underlying undirected graph is connected.

A $\left\{ \begin{array}{l} \text{path} \\ \text{cycle} \end{array} \right\}$ in a directed graph is a sequence of nodes that forms

a $\left\{ \begin{array}{l} \text{path} \\ \text{cycle} \end{array} \right\}$ in the underlying undirected graph plus a sequence of connecting arcs between these nodes.

Exception: cycles of length 2 are allowed!

A path (or cycle) can have forward and backward edges.

A path with only forward arcs is called directed.

Trees. A tree is an undirected graph that is connected and has no cycles. A node of degree 1 in a tree is called a leaf.

Theorem.

- Every tree with more than one node has at least one leaf.
(in fact: two leaves!)
- An undirected graph with n nodes is a tree if and only if it is connected and has $n-1$ edges.
- Given two nodes u, v in a tree, $u \neq v$, there exists a unique path from u to v .
- If an undirected graph G arises from a tree by adding one new edge, then G has exactly one cycle.

Let $G = (V, E)$ be a connected undirected graph. A tree $T = (V, E')$ with $E' \subseteq E$ is called a spanning tree.

Theorem. Let $G = (V, E)$ be a connected undirected graph and $E_0 \subseteq E$ a set of edges that forms no cycles. Then there exists a set $E' \subseteq E$, $E_0 \subseteq E'$ such that $T = (V, E')$ is a spanning tree of G .

7.2 The network flow problem

A network is a directed graph $G = (V, E)$ together with additional numerical information:

- $b_v \in \mathbb{R} \quad \forall v \in V$ (inflow at node v)
- $u_e \geq 0$ or $+\infty \quad \forall e \in E$ (capacity of edge e)
- $c_e \in \mathbb{R} \quad \forall e \in E$ (cost of edge e)

A node with $b_v > 0$ is called a source and a sink if $b_v < 0$.

In the network flow problem we are searching for a flow $(f_e)_{e \in E} \in \mathbb{R}^E$.
A vector $(f_e)_{e \in E}$ is a feasible flow if these constraints are satisfied:

$$\sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e = b_v \quad \forall v \in V \quad (\text{flow conservation})$$

$$f_e \leq u_e \quad \forall e \in E$$

$$f_e \geq 0 \quad \forall e \in E$$

Note:
$$\sum_{v \in V} \left(\sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e \right) \stackrel{!}{=} \sum_{e \in E} f_e - f_e = 0$$

|| \leftarrow flow conservation

$$\sum_{v \in V} b_v$$

The general minimal cost network flow problem:

$$\min \sum_{e \in E} c_e f_e$$

$$\text{s.t.} \quad \sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e = b_v \quad \forall v \in V$$


$$\left. \begin{array}{l} f_e \leq u_e \\ f_e \geq 0 \end{array} \right\} \forall e \in E$$

A linear program in standard form if $u_e = \infty \forall e \in E$.
(uncapacitated network)

Special cases:

- shortest path
- max. flow
- transportation problem (equivalent!)
- assignment problem

Variants (equivalent)

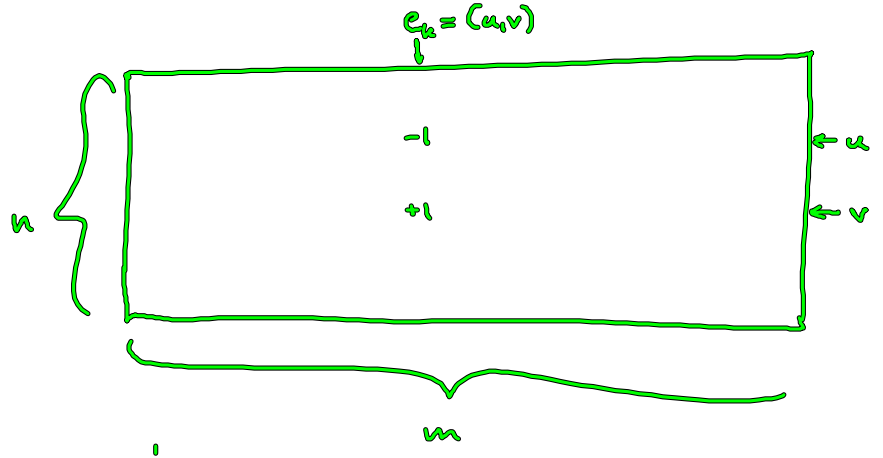
- transportation problem
- problem with only one source and one sink
-  (circulation problem)
- network with node capacities
- network with lower bounds on the edge flows: $f_e \geq d_e$

Matrix formulation.

W.l.o.g. assume $V = \{1, \dots, n\}$ and $|E| = m \geq n-1$, edges somehow ordered: $E = \{e_1, \dots, e_m\}$

Incidence matrix: $A = (a_{i,k})_{\substack{1 \leq i \leq n \\ 1 \leq k \leq m}}$ with

$$a_{i,k} := \begin{cases} +1 & \text{if } e_k = (i, v) \text{ for some } v \in V \\ -1 & \text{if } e_k = (v, i) \text{ } \\ 0 & \text{otherwise} \end{cases}$$



If $f = (f_{e_1}, \dots, f_{e_m})^T$ is a flow vector then flow conservation states

$$A \cdot f = b$$

The rows of A add up to the zero vector $\Rightarrow \text{rank } A \leq n - 1$

A flow vector f with $A \cdot f = 0$ is called a circulation.