

### 3.3 The potential reduction algorithm

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & p^T b \\ \text{s.t.} \quad & p^T A + s^T = c^T \\ & s \geq 0 \end{aligned}$$

Assume that rows of  $A$  are linearly independent and there exist  $x^0 > 0$  and  $(p^0, s^0)$  with  $s^0 > 0$  which are feasible for primal and dual LP, respectively.

Main new idea: Stay away from the boundary.

Potential function:

$$G(x, s) = q \cdot \log s^T x - \sum_{j=1}^n \log x_j - \sum_{j=1}^n \log s_j$$

where  $q > n$  is a constant.

If  $x$  and  $(p, s)$  are feasible, then

$$c^T x - b^T \cdot p = (s^T + p^T A) \cdot x - x^T \cdot A^T \cdot p = s^T x \quad \text{"duality gap"}$$

Theorem: An algorithm that maintains primal and dual feasibility and reduces  $G(x, s)$  by at least  $\delta > 0$  at each iteration, finds solutions with duality gap  $(s^k)^T \cdot x^k \leq \varepsilon$  after

$$K = \left\lceil \frac{G(x^0, s^0) + (q-u) \cdot \log \frac{1}{\varepsilon} - u \cdot \log u}{\delta} \right\rceil$$

iterations.

Proof:

$$G(x, s) = q \cdot \log s^T x - \sum_{j=1}^n \log x_j - \sum_{j=1}^n \log s_j$$

$$= u \cdot \log s^T x - \underbrace{\sum_{j=1}^n \log(x_j \cdot s_j)}_{\text{minimum attained for } x_j \cdot s_j = \frac{s^T \cdot x}{u}} + (q-u) \cdot \log s^T x$$

$$\geq u \cdot \log u + (q-u) \cdot \log s^T x$$

$$\Rightarrow u \cdot \log s^T x - \sum_{j=1}^n \log x_j - \sum_{j=1}^n \log s_j \geq u \cdot \log u \quad (*)$$

Suppose that  $G(x^{k+1}, s^{k+1}) - G(x^k, s^k) \leq -\delta \quad \forall k$

$$\Rightarrow G(x^k, s^k) - G(x^0, s^0) \leq -K \cdot \delta$$

Our choice of  $K$  yields:

$$G(x^k, s^k) \leq -(q-u) \cdot \log \frac{1}{\varepsilon} + u \log u$$

Using def. of  $G(x, s)$  and  $(*)$  yields:

$$G(x^k, s^k) \geq u \cdot \log u + (q-u) \cdot \log (s^k)^T \cdot x^k$$

$$\Rightarrow (s^k)^T \cdot x^k \leq \varepsilon.$$

□

→ "potential reduction" algorithm.

Idea: Starting with feasible  $x > 0$  and  $(p, s)$  with  $s > 0$ , try to find direction  $d$  such that

$$G(x+d, s) < G(x, s)$$

such that  $A \cdot d = 0$ ,  $\|X^{-1} \cdot d\| \leq \beta < 1$ .

Minimizing  $G(x+d, s)$  s.t.  $A \cdot d = 0$ ,  $\|X^{-1} \cdot d\| \leq \beta$  is difficult due to objective function.

Idea: Linearize the objective function by taking the first order Taylor series expansion in  $d$ .

$$\min (\nabla_x G(x, s))^T \cdot d$$

$$\text{s.t. } A \cdot d = 0$$

$$\|X^{-1} \cdot d\| \leq \beta$$

(Same result as in 3.1. except for different objective function  $\hat{c}$ )

$$\hat{c}_i = \frac{\partial G(x, s)}{\partial x_i} = \frac{q \cdot s_i}{s^T x} - \frac{1}{x_i}$$

Applying lemma from Sect. 3.1, we obtain opt. sol.

$$d^* = -\beta \cdot X \cdot \frac{u}{\|u\|}$$

where

$$u = X \cdot (\hat{c} - A^T \cdot (A X^2 A^T)^{-1} A X^2 \hat{c})$$

Since  $X \cdot \hat{c} = \frac{q}{s^T X} X \cdot s - \underline{\Pi}$ , we get

$$u = (\mathbb{I} - X A^T (A X^2 A^T)^{-1} A X) \left( \frac{q}{s^T X} X s - \underline{\Pi} \right)$$

Moreover,  $G(x, s)$  decreases by  $\beta \cdot \|u\| + \mathcal{O}(\beta^2)$   
 $\uparrow$  Lemma in 9.1.  $\uparrow$  higher order terms.

$\Rightarrow$  If  $\|u\|$  is large enough, then  $G(x, s)$  decreases by at least  $\delta > 0$ . Otherwise update dual variables (omit details).

### The potential reduction algorithm

1) Start with feasible  $x^0 > 0$ ,  $(p^0, s^0)$  with  $s^0 > 0$ ;  
set  $k := 0$

2) If  $(s^k)^T \cdot x^k < \varepsilon$  stop.

3) Let  $X_k = \text{diag}(x_1^k, \dots, x_n^k)$

$$\bar{A}_k = (A \cdot X_k)^T (A X_k^2 A^T)^{-1} \cdot A \cdot X_k$$

$$u^k = (\mathbb{I} - \bar{A}_k) \cdot \left( \frac{q}{(s^k)^T X_k} X_k s - \underline{\Pi} \right)$$

$$d^k = -\beta \cdot X_k \cdot u^k / \|u^k\|.$$

4) (Primal step) If  $\|u^k\| \geq \gamma$ , then

$$x^{k+1} := x^k + d^k, \quad s^{k+1} = s^k, \quad p^{k+1} = p^k$$

5) (Dual step) If  $\|u^k\| < \gamma$ , then

$$x^{k+1} = x^k, \quad s^{k+1} = \frac{(s^k)^T \cdot x^k}{q} (X_k)^{-1} \cdot (u^k + \mathbb{I})$$

$$p^{k+1} = p^k + (A X_k^2 A^T)^{-1} A X_k (X_k s - \frac{(s^k)^T x^k}{q} \cdot \mathbb{I})$$

6)  $k := k+1$ , goto 2.

Theorem: Let  $\beta < 1$ ,  $\gamma < 1$

(i) If  $\|u^k\| \geq \gamma$  then

$$G(x^{k+1}, s^{k+1}) - G(x^k, s^k) \leq -\beta \cdot \gamma + \frac{\beta^2}{2(1-\beta)}$$

(ii) If  $\|u^k\| < \gamma$  then

$$G(x^{k+1}, s^{k+1}) - G(x^k, s^k) \leq -(\gamma - \eta) + \eta \cdot \log \frac{q}{\eta} + \frac{\gamma^2}{2(1-\gamma)}$$

(iii) If  $q = \eta + \sqrt{\eta}$ ,  $\beta \approx 0.285$ ,

$\gamma \approx 0.473$ , then  $G(x, s)$  decreases by at least  $\delta = 0.073$  at each iteration.

Proof: see book. □

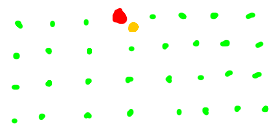
Initialization: see book (auxiliary problem....)

Complexity of the algorithm:

The potential reduction algorithm finds  $\varepsilon$ -opt. solutions in time

$$O(u^{3.5} \cdot \log^{1/\varepsilon} + u^5 \cdot \log(u \cdot U))$$

where  $U = \max\{|a_{ij}|, |b_i|, |c_j|\}$  (all integer)



If  $\varepsilon$  is taken sufficiently small, an optimal solution can be found by rounding.