

Primal path following algorithm

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & p^T b \\ \text{s.t.} \quad & p^T A + s^T = c^T \\ & s \geq 0 \end{aligned}$$

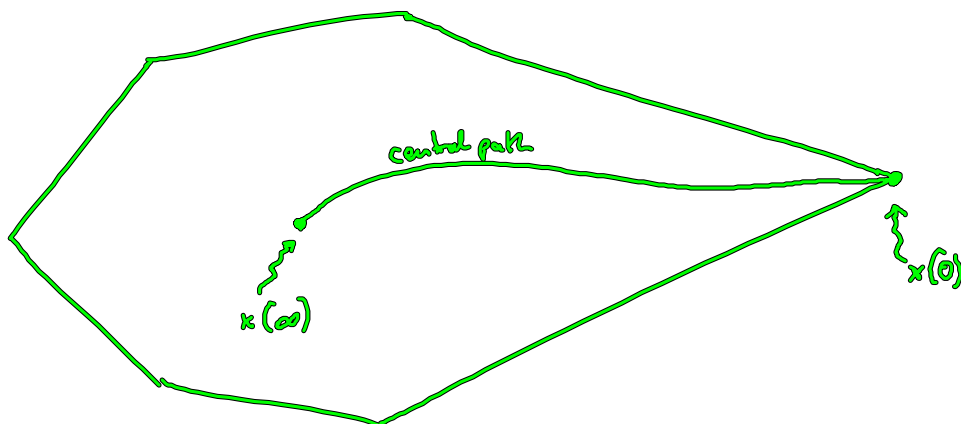
► Step 1: Ignore the inequality constraints and modify the objective function \rightarrow barrier function: $\mu > 0$

$$B_\mu(x) := \begin{cases} c^T x - \mu \sum_{i=1}^n \log x_i & \text{for } x > 0 \\ \infty & \text{if } x_i \leq 0 \text{ for some } i \end{cases}$$

\rightarrow barrier problem:
$$\begin{aligned} \min \quad & B_\mu(x) \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

Assume all barrier problems for $\mu > 0$ have an optimal solution $x(\mu)$.

The central path is the set $\{x(\mu) \mid \mu > 0\}$



• $\lim_{\mu \rightarrow 0} x(\mu)$ is the optimal solution to the original LP

• $\lim_{\mu \rightarrow \infty} x(\mu)$ is called the analytic center

Ex: ① $\min x$
s.t. $x \geq 0$

→ barrier function: $B_{\mu}(x) = x - \mu \log x \quad (x > 0)$

→ minimize: $\frac{d}{dx} B_{\mu}(x) = 1 - \frac{\mu}{x} = 0 \Leftrightarrow x = \mu$

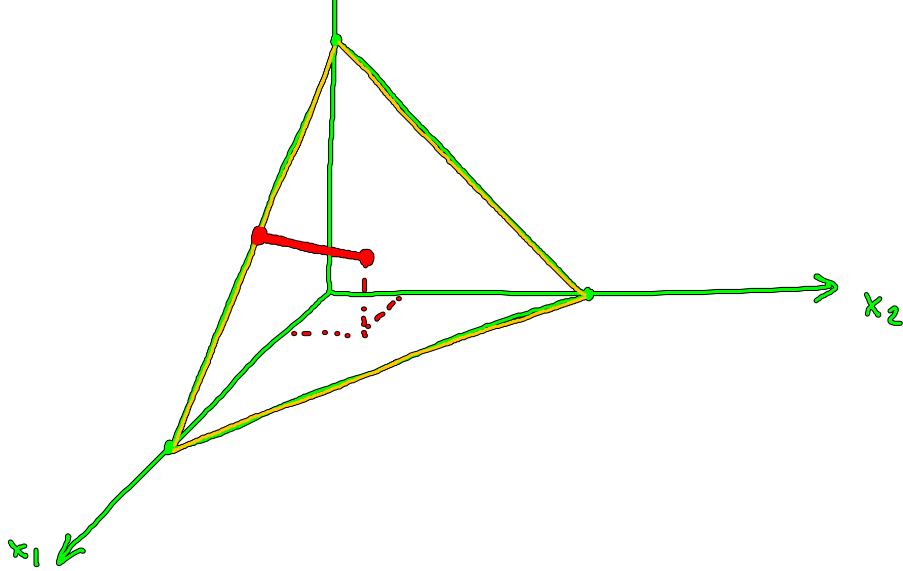
→ $x(\mu) = \mu$

→ $\lim_{\mu \rightarrow 0} x(\mu) = 0$

② $\min x_2$
s.t. $x_1 + x_2 + x_3 = 1$
 $x_i \geq 0$

→ barrier problem: $\min x_2 - \mu \log x_1 - \mu \log x_2 - \mu \log x_3$
s.t. $x_1 + x_2 + x_3 = 1$

→ central path: $x(\mu) = \frac{1}{2} \begin{pmatrix} \sqrt{9\mu^2 + 2\mu + 1} - 3\mu \\ 2 + 6\mu - 2\sqrt{9\mu^2 + 2\mu + 1} \\ \sqrt{9\mu^2 + 2\mu + 1} - 3\mu \end{pmatrix}$
 $x_3 \uparrow$



$$\lim_{\mu \rightarrow 0} x(\mu) = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix}$$

$$\lim_{\mu \rightarrow \infty} x(\mu) = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

Dual barrier problem: $\max p^T b + \mu \sum_{i=1}^n \log s_i$
 s.t. $p^T A + s^T = c^T$

Lemma. (Karush-Kuhn-Tucker conditions)

$x(\mu)$, $p(\mu)$, $s(\mu)$ are optimal solutions to the primal and dual barrier problems w.r.t. μ if and only if the following conditions hold:

$$\begin{aligned} A x(\mu) &= b \\ x(\mu) &\geq 0 \end{aligned}$$

$$\begin{aligned} A^T p(\mu) + s(\mu) &= c \\ s(\mu) &\geq 0 \end{aligned}$$

$$X(\mu) S(\mu) e = \mu e$$

with $X(\mu) = \begin{pmatrix} x_1(\mu) & & \\ & \ddots & \\ & & x_n(\mu) \end{pmatrix}$

$$S(\mu) = \begin{pmatrix} s_1(\mu) & & \\ & \ddots & \\ & & s_n(\mu) \end{pmatrix}$$

$$e = (1, \dots, 1)$$

$$x_i(\mu) \cdot s_i(\mu) = \mu \quad \forall i$$

► Step 2: Approximate the barrier function by its Taylor series:

$$\begin{aligned} B_\mu(x+d) &\approx B_\mu(x) + \sum_{i=1}^n \left(\frac{\partial}{\partial x_i} B_\mu(x) \right) \cdot d_i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \left(\frac{\partial^2}{\partial x_i \partial x_j} B_\mu(x) \right) \cdot d_i \cdot d_j \\ &= B_\mu(x) + (c^T - \mu e^T X^{-1}) \cdot d + \frac{1}{2} \mu d^T \cdot (X^2)^{-1} \cdot d \end{aligned}$$

$\leftarrow c_i - \frac{\mu}{x_i}$
 $\leftarrow 0$ if $i \neq j$
 $\leftarrow \frac{\mu}{x_i^2}$ if $i = j$

$$\begin{aligned} \rightarrow \min & (c^T - \mu e^T X^{-1}) \cdot d + \frac{1}{2} \mu d^T \cdot (X^2)^{-1} \cdot d \\ \text{s.t.} & A \cdot d = 0 \end{aligned}$$

► Step 3: Solve the approximation with Lagrangean multipliers:

Lagrangean function:

$$L(d, p) = (c^T - \mu e^T X^{-1}) d + \frac{1}{2} \mu d^T (X^2)^{-1} d - p^T \cdot A d$$

$$\text{with } \frac{\partial}{\partial d_i} L(d, p) = 0 \quad \forall i$$

$$\frac{\partial}{\partial p_i} L(d, p) = 0 \quad \forall i$$

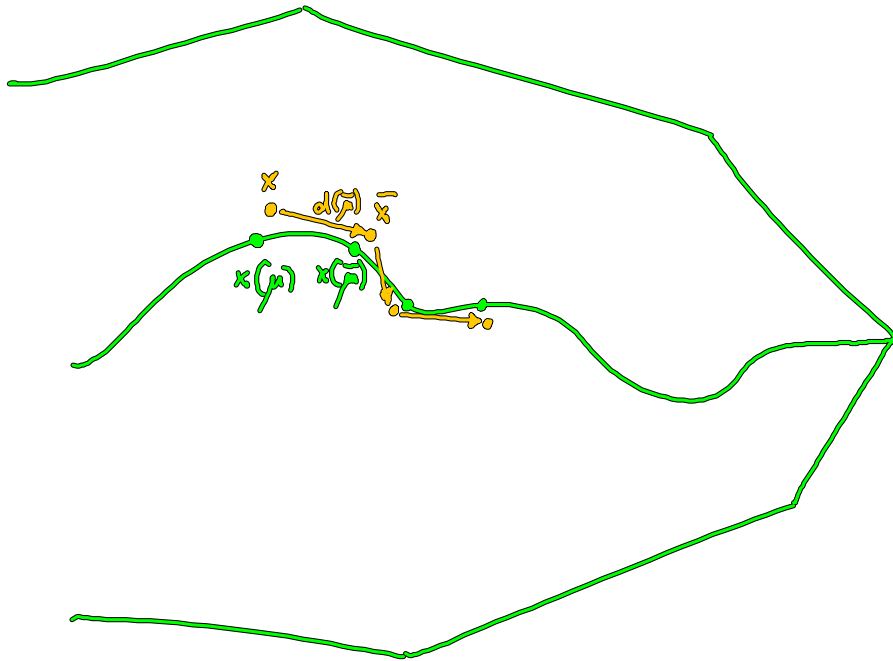
→ leads to a linear system:

$$\begin{pmatrix} \mu(X^2)^{-1} & -A^T \\ A & 0 \end{pmatrix} \cdot \begin{pmatrix} d \\ p \end{pmatrix} = \begin{pmatrix} \mu X^{-1} e - c \\ 0 \end{pmatrix}$$

solving this equation for d and p leads to

$d(\mu) \leftarrow$ Newton direction
 $p(\mu)$

Newton step: given x, p, s, μ
 new primal solution: $\bar{x} = x + d(\mu)$
 new dual solution: $p(\mu), c - A^T p(\mu)$
 decrease μ to $\bar{\mu} = \alpha \mu$ with $0 < \alpha < 1$ (fixed)



► Algorithm:

Input: $A, b, c, \varepsilon > 0$ (optimality tolerance), $0 < \alpha < 1, \mu^0$
 initial primal/dual feasible solutions $x^0 > 0, s^0 > 0, p^0$

1. $k := 0$

2. If $(s^k)^T x^k < \varepsilon$ then STOP: current solution is ε -optimal

3.

$$X_k := \begin{pmatrix} x_1^k & & \\ & \ddots & \\ & & x_n^k \end{pmatrix}, \quad \mu^{k+1} := \alpha \mu^k$$

4. Solve the linear system

$$\begin{pmatrix} \mu^{k+1} (X_k^2)^{-1} & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} d \\ p \end{pmatrix} = \begin{pmatrix} \mu^{k+1} X_k^{-1} e - c \\ 0 \end{pmatrix}$$

for d and p

5. $x^{k+1} := x^k + d$, $p^{k+1} := p$, $s^{k+1} := c - A^T p$

6. $k := k + 1$, go to 2.