Theorem. Let $0 < \beta < 1$, $\mu^0 > 0$, $\varepsilon > 0$.

If the primal path following method is used with parameter $\alpha = 1 - \frac{\sqrt{\beta} - \beta}{\sqrt{\beta} + \sqrt{n}}$ and initial primal/dual feasible solution $x^0 > 0$, $s^0 > 0$, p^0 that satisfy

$$\left\| \frac{1}{\mu^0} X_0 S_0 e - e \right\| \leq \beta,$$

then a primal/dual feasible solution (x^K, p^K, s^K) with duality gap

$(s^K)^T x^K \leq \varepsilon$ is found after

$$K := \left\lceil \frac{\sqrt{\beta} + \sqrt{n}}{\sqrt{\beta} - \beta} \log\left(\frac{(s^0)^T x^0}{\varepsilon} \cdot \frac{1+\beta}{1-\beta} \right) \right\rceil = \left\lceil \frac{1}{1-\alpha} \log\left(\frac{\varepsilon_0}{\varepsilon} \cdot \frac{1+\beta}{1-\beta} \right) \right\rceil$$

iterations.

$\longrightarrow$ solve an auxiliary LP with μ^0, x^0, s^0 such that

$$\left\| \frac{1}{\mu^0} X_0 S_0 e - e \right\| = \frac{1}{4}$$

$\Rightarrow$ # of iterations to reduce the duality gap from $\varepsilon_0 = (s^0)^T x^0$ to ε is

$$\left\lceil \left(2 + \tfrac{1}{4}\sqrt{n}\right) \log\left(\frac{\varepsilon_0}{\varepsilon} \cdot \frac{5}{3} \right) \right\rceil = O\left(\sqrt{n} \log \frac{\varepsilon_0}{\varepsilon} \right)$$

$\rightarrow$ polynomial in n

with $O((m+n)^3)$ operations in one iteration

$$\begin{array}{c} O((m+n)^3) \\ \parallel \\ O(n^3) \end{array}$$

Primal-dual path following algorithm

Idea: Directly try to solve the KKT conditions:

$$\begin{aligned} Ax(\mu) &= b \\ x(\mu) &\geq 0 \\ A^T p(\mu) + s(\mu) &= c \\ s(\mu) &\geq 0 \\ X(\mu)S(\mu)e &= e\mu \end{aligned}$$

- Newton's method:

Given $F: \mathbb{R}^r \longrightarrow \mathbb{R}^r$, problem: find a z^* such that $F(z^*) = 0$

Assume we have an approximation z for z^*

$\longrightarrow$ find a direction d such that $z + d$ is a better approx.

Taylor series: $F(z+d) \approx F(z) + J_F(z) \cdot d$

with the Jacobean $J_F(z) = \left(\frac{\partial}{\partial z_j} F_i(z) \right)_{1 \le i,j \le n}$

$$F(z+d) = 0 \iff F(z) + J_F(z) \cdot d = 0$$

- Here:

$$F(x,p,s) = \begin{pmatrix} Ax - b \\ A^T p + s - c \\ XSe - \mu e \end{pmatrix}$$

$\rightarrow$ solve the linear system

$$\boxed{\begin{aligned} \begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{pmatrix} \cdot \begin{pmatrix} d_x \\ d_p \\ d_s \end{pmatrix} &= -F(x,p,s) \\ (*) \qquad &= \begin{pmatrix} 0 \\ 0 \\ \mu e - XSe \end{pmatrix} \end{aligned}}$$

- Updating the solution and the barrier parameter:

new solution: $x + \beta_{primal} d_x$, $p + \beta_{dual} d_p$, $s + \beta_{dual} d_s$

with
$$\beta_{\text{primal}} = \min \left\{ 1, \ \alpha \cdot \min_{\substack{i \\ \text{such that} \\ (d_x)_i < 0}} \left(-\frac{x_i}{(d_x)_i} \right) \right\}$$

$$\beta_{\text{dual}} = \min \left\{ 1, \ \alpha \cdot \min_{\substack{i \\ \text{such that} \\ (d_s)_i < 0}} \left(-\frac{s_i}{(d_s)_i} \right) \right\} \qquad (**)$$

where $0 < \alpha < 1$.

$$\bar{\mu} := \rho \cdot \frac{s^T x}{n} \qquad \text{with} \qquad 0 < \rho \leq 1 \qquad (***)$$

- Algorithm:

Input: $A, b, c, \ \varepsilon > 0, \ 0 < \alpha < 1, \ \mu^0$

initial primal & dual feasible solutions $x^0 > 0$, $s^0 > 0$, p^0

1. $k := 0$
2. If $(s^k)^T x^k < \varepsilon$ then STOP: solution is ε-optimal.
3. Let $\rho^k \in (0, 1]$, update μ^k according to $(***)$.
   Solve the linear system $(*)$ for d_x^k, d_p^k, d_s^k
4. Find the step lengths β_{primal} and β_{dual} according to $(**)$
5. $x^{k+1} := x^k + \beta_{\text{primal}} \cdot d_x^k$,
   $p^{k+1} := p^k + \beta_{\text{dual}} \cdot d_p^k$, $s^{k+1} := s^k + \beta_{\text{dual}} \cdot d_s^k$
6. $k := k+1$, go to 2.

- Complexity: $O(\sqrt{n} \log \frac{\varepsilon_0}{\varepsilon})$ iterations,
  (even $O(\log n \log \frac{\varepsilon_0}{\varepsilon})$ on average ?)

- Infeasible primal-dual path following method:
  Start with $x^0 > 0$, $s^0 > 0$, p^0 that do not necessarily have to be feasible.
  Only change: instead of the system $(*)$ solve:

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S_k & 0 & X_k \end{pmatrix} \cdot \begin{pmatrix} d_x^k \\ d_p^k \\ d_s^k \end{pmatrix} = - \begin{pmatrix} Ax^k - b \\ A^T p^k + s^k - c \\ X_k S_k e - \mu^k e \end{pmatrix}.$$

- Summary:
  Interior point methods start with an interior point of the feasible set and try to reach a primal/dual feasible solution pair with duality gap $< \varepsilon$ by proceeding in a certain direction d_{method} in every step.

$$d_{\text{affine scaling}} = -X^2\left(I - A^T(AX^2A^T)^{-1}AX^2\right)c$$

$$d_{\text{path following}} = \left(I - A^T(AX^2A^T)^{-1}AX^2\right)\left(Xe - \frac{1}{\mu}X^2c\right)$$

Let $d_{\text{center}} := \left(I - A^T(AX^2A^T)^{-1}AX^2\right)Xe$, then

$$d_{\text{path following}} = d_{\text{center}} + \frac{1}{\mu} d_{\text{affine scaling}}$$

$$d_{\text{potential reduction}} = d_{\text{center}} + \frac{q}{s^T x} d_{\text{affine scaling}}$$

- Comparison with the simplex algorithm:
  - in practise interior point methods tend to have a better performance for:
    - very large problems
    - massively degenerate problems
  - interior point methods always find optimal solutions in the interior of the set of all optimal solutions (not a basic optimal solution)
    - → IP
    - → sensitive for unbounded sets of optimal solutions
    - → problems with solving many similar problems