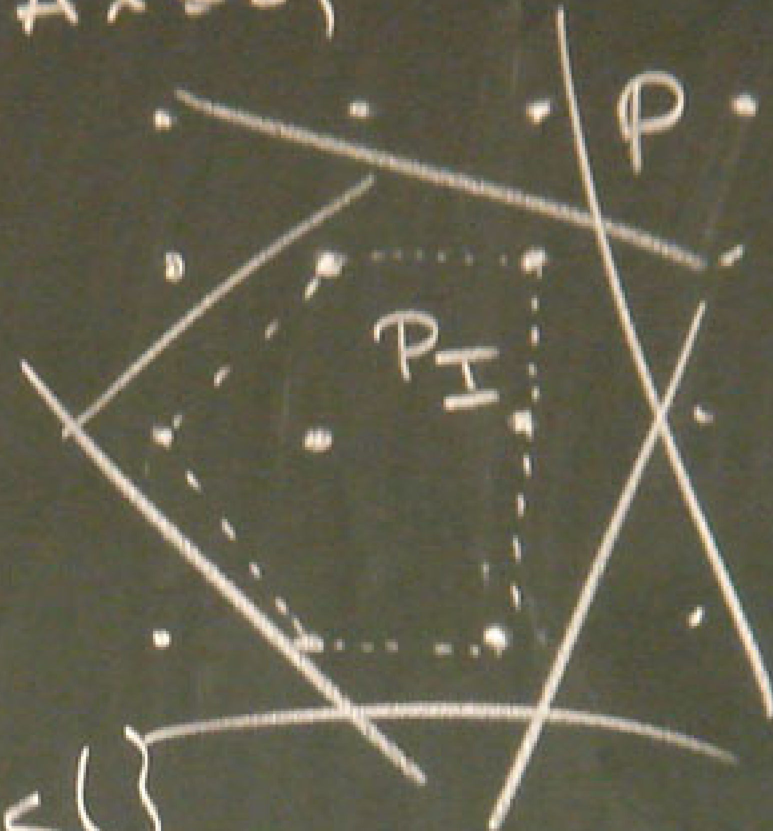
$$\max c^T x$$
$$\text{s.t. } Ax \le b$$
$$x \text{ integer}$$
$$x \in \mathbb{Z}^n$$

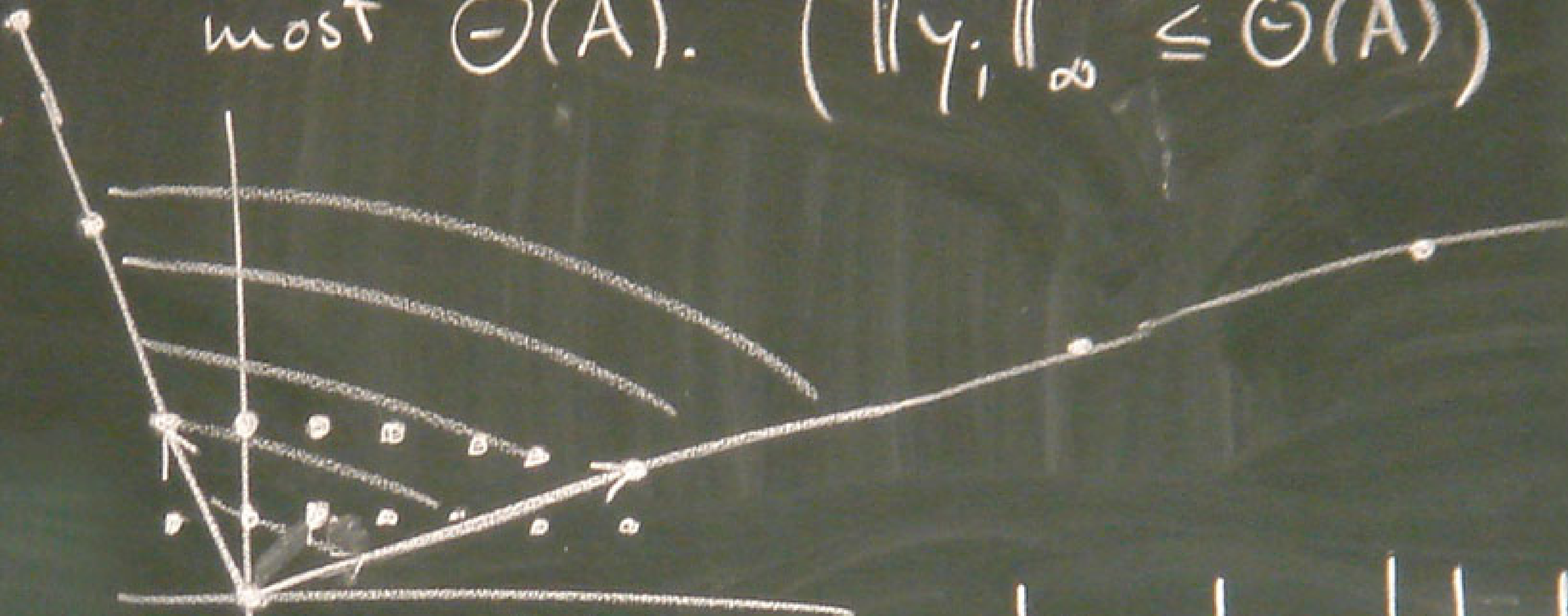
$P = \{x \mid Ax \le b\}$



$P_I = \operatorname{conv}\{x \in \mathbb{Z}^n \mid Ax \le b\}$

<u>Lemma:</u> $A \in \mathbb{Z}^{m \times n}$ and $C = \{x \mid Ax \ge 0\}$
C is generated by a finite set of integral vectors y_i each having components

with absolute value at most $\Theta(A)$. $\left(\|y_i\|_\infty \leq \Theta(A) \right)$



Lemma 5.4: Each rational polyhedral cone C is generated by a finite set of integral vectors $\{a_1, \ldots, a_t\}$ such that each integral vector x in C is

a non-negative integral comb of $a_1, \dots, a_t$, i.e.

$$x = \sum_{i=1}^{t} \lambda_i \cdot a_i \qquad \lambda_i \geq 0, \ \lambda_i \in \mathbb{Z}$$

("Hilbert basis" of C)

<u>Proof:</u> C is generated by integral vectors $b_1, \dots, b_k$. Let $a_1, \dots, a_t$ be all integral vectors in the polytope

$$P = \{\lambda_1 b_1 + \dots + \lambda_k b_k \mid 0 \le \lambda_i \le 1 \text{ for } i=1,\dots,k\}$$

Since $\{b_1,\dots,b_k\} \subseteq \{a_1,\dots,a_t\}$, the a_i's generate C.

Let $x \in C \cap \mathbb{Z}^n$

$$x = \sum_{i=1}^{k} \mu_i b_i \qquad \mu_i \ge 0$$

$$= \sum_{i=1}^{k} \lfloor \mu_i \rfloor \cdot b_i + \underbrace{\sum_{i=1}^{k} (\mu_i - \lfloor \mu_i \rfloor) \cdot b_i}_{\substack{\in P \cap \mathbb{Z}^n \\ = a_j}}$$

Theorem: $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{R}^n$,
$P = \{x \mid Ax \leq b\}$ and $P_I \neq \emptyset$.

(a) If y is opt. sol. of $\max c^T x$ s.t. $x \in P$ then there is an opt. integral sol z of $\max c^T x$ s.t. $x \in P_I$ with

$$\|z - y\|_\infty \leq n \cdot \Theta(A)$$

(b) If y is a feasible integral but not opt. solution to $\max c^T x$ s.t. $x \in P_I$, then there is $z \in P_I \cap \mathbb{Z}^n$ with $c^T z > c^T y$ and $\|z - y\|_\infty \leq n \cdot \Theta(A)$

<u>Proof</u>. Let $y \in P$ arbitrary and z^* opt. integral solution of $\max c^T x$ s.t. $x \in P_I$. Split $Ax \leq b$ into two subsystems

$$\begin{array}{lcl} A_1 x \leq b_1 & \text{with} & A_1 z^* \geq A_1 y \\ A_2 x \leq b_2 & & A_2 z^* < A_2 y \end{array}$$

$\Rightarrow \underline{z^* - y} \in C = \{x \mid A_1 x \geq 0, A_2 x \leq 0\}$

C is generated by $x_1, \dots, x_s \in \mathbb{Z}^n$ with $\|x_i\|_\infty \leq \Theta(A)$

$$\Rightarrow z^* - y = \sum_{i=1}^{s} \lambda_i x_i \quad \text{for } \lambda_i \geq 0$$

with at most n of the λ_i's positive (Carathéodory).

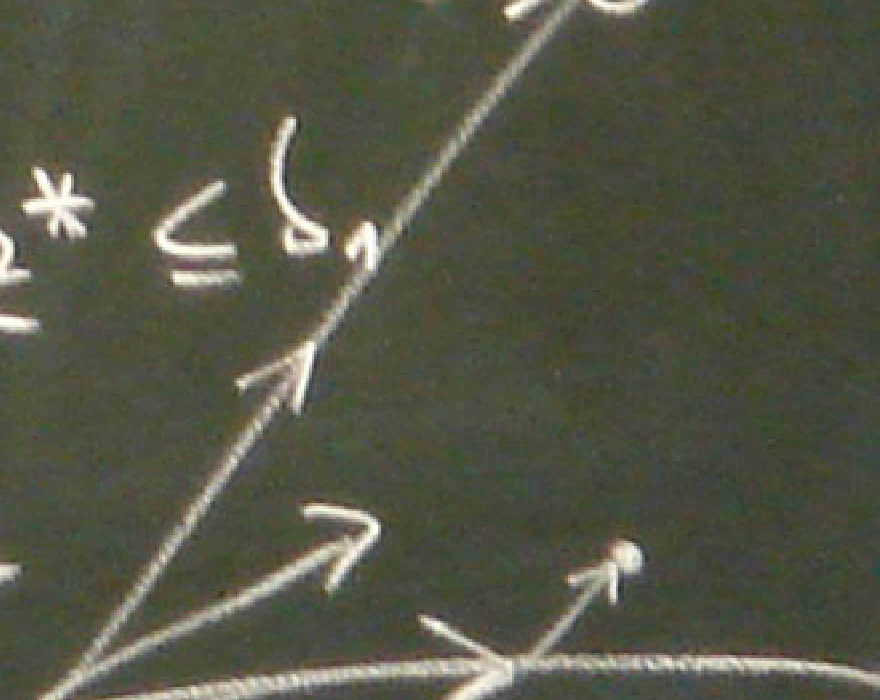
For $\mu = (\mu_1, \dots, \mu_s)$ with $0 \le \mu_i \le \lambda_i$ define

$$z_\mu := z^* - \sum_{i=1}^{s} \underbrace{\mu_i}_{\geq 0} x_i = y + \sum_{i=1}^{s} \underbrace{(\lambda_i - \mu_i)}_{\geq 0} \cdot x_i$$

Then: $A_1 \cdot z_\mu \le A_1 \cdot z^* \le b_1$

$A_2 z_\mu \le A_2 \cdot y \le b_2$

$\Rightarrow z_\mu \in P$.



<u>Case 1:</u> $\exists\, i \in \{1, \dots, s\} : \lambda_i \geq 1$ and $c^T x_i > 0$.

Let $z := y + x_i \Rightarrow z \in P$ and $c^T z > c^T y$ ($\lightning$ in (a))

For (b): y integral $\Rightarrow z$ integral and $\|z - y\|_\infty = \|x_i\|_\infty \leq \Theta(A)$.

Case 2: $\forall i \in \{1,\dots,s\} : \lambda_i \geq 1 \Rightarrow c^T x_i \leq 0$

Let $z := z_{\lfloor \lambda \rfloor} = z^* - \sum_{i=1}^{s} \lfloor \lambda_i \rfloor x_i \in P \cap \mathbb{Z}^n$

and $c^T z \geq c^T z^*$ ($\Rightarrow z$ opt. ILP solution)

and $\|z - y\|_\infty = \| \sum_{i=1}^{s} (\lambda_i - \lfloor \lambda_i \rfloor) \cdot x_i \|_\infty \leq \sum_{i=1}^{s} \underbrace{(\lambda_i - \lfloor \lambda_i \rfloor)}_{\leq 1} \cdot \underbrace{\|x_i\|_\infty}_{\leq \Theta(A)} \leq n \cdot \Theta(A)$

$\square$

Corollary: An opt sol to an ILP can be encoded polynomially in the input size.

$\Longrightarrow$ Integer linear inequalities is in NP.

By part (b) of the theorem above, optimality of $y \in P_I \cap \mathbb{Z}^n$ can be checked by testing $y+x$ for a finite set of vectors X that depend only on A. ("test set")

Test sets are sometimes useful in practice.

Theorem: For each $A \in \mathbb{Z}^{m \times n}$ there exists $M \in \mathbb{Z}^{m' \times n}$ (with $|M_{ij}| \leq n^{2n} \Theta(A)^n$) such that for each $b \in \mathbb{Q}^m$ there is $d \in \mathbb{Q}^{m'}$ with:

$$P_I = \{x \mid Ax \leq b\}_I = \{x \mid Mx \leq d\}$$