

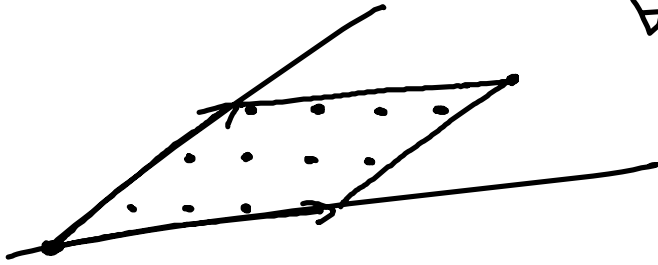
$$C = \{x \mid Ax \geq 0\} \quad A \in \mathbb{Q}^{m \times n}$$

$\rightarrow \exists$ Hilbert basis $\{a_1, \dots, a_t\}$:

$$C = \text{cone}(\{a_1, \dots, a_t\}) \text{ and}$$

$$\forall x \in C \cap \mathbb{Z}^n:$$

$$x = \sum_{i=1}^t \lambda_i \cdot a_i \text{ with } \lambda_i \geq 0 \\ \lambda_i \in \mathbb{Z}$$



x opt. LP solution

y opt ILP solution $\|x - y\|_\infty \leq u \cdot \Theta(A)$

$$P = \{x \mid Ax \leq b\} \quad \max c^T x \text{ s.t. } x \in P \cap \mathbb{Z}^n$$

y_1 feasible ILP solution, not optimal

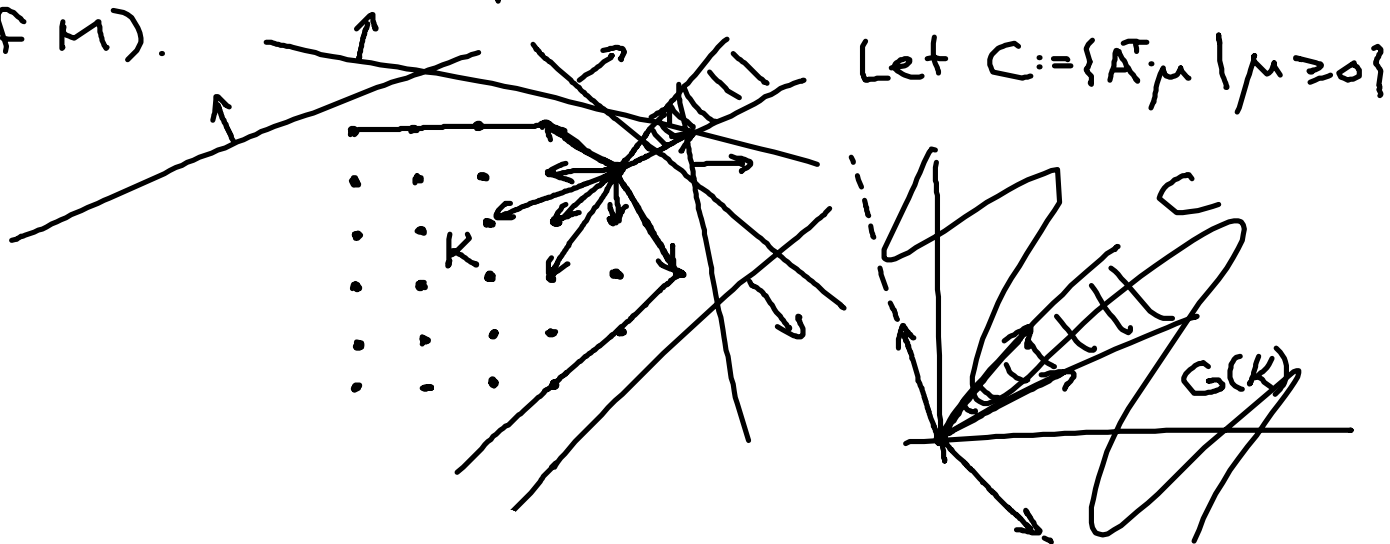
$\rightarrow \exists y_2$ feasible ILP solution with $c^T y_2 > c^T y_1$

and $\|y_1 - y_2\|_\infty \leq u \cdot \Theta(A)$

Theorem: For each $A \in \mathbb{Z}^{m \times n}$ there exists $M \in \mathbb{Z}^{m' \times n}$ (with $|M_{ij}| \leq u^{2n} \cdot \Theta(A)^2$) such that for each $b \in \mathbb{Q}^m$ there is $d \in \mathbb{Q}^{m'}$ with

$$P_I = \{x \mid Ax \leq b\}_I = \{x \mid M \cdot x \leq d\}.$$

Proof: (we don't prove the bound on the entries of M).

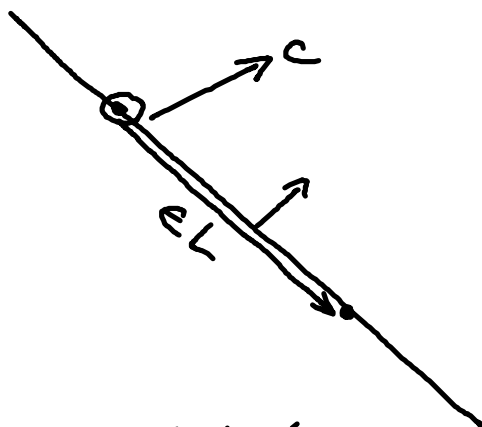


$$L := \{z \in \mathbb{Z}^n \mid \|z\|_\infty \leq u \cdot \Theta(A)\} \quad \text{"test set"}$$

For $K \subseteq L$ let $C_K := C \cap \{y \mid z^T \cdot y \leq 0 \forall z \in K\}$

C_K is a polyhedral cone (note that C is finitely generated) and therefore generated by a finite set of integer vectors $G(K) \subseteq \mathbb{Z}^n$.

Let $M \in \mathbb{Z}^{m' \times n}$ with rows $\bigcup_{K \subseteq L} G(K)$.



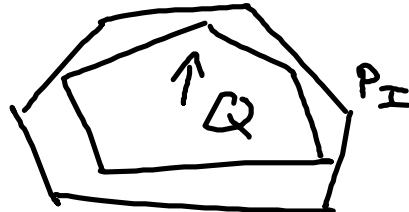
Assume that $P_I \neq \emptyset$ (otherwise see book).

and let $\delta_y := \max \{y^T \cdot x \mid Ax \leq b, x \in \mathbb{Z}^n\}$ for all $y \in \bigcup_{K \in L} G(K)$

We will show that $P_I = \{x \mid y^T x \leq \delta_y \forall y \in \bigcup_{K \in L} G(K)\}$
 $= \{x \mid M \cdot x \leq \delta\}$

$P_I \subseteq Q$: clear by definition of $Q :=$ of δ_y .

$P_I \supseteq Q$:



Show that

$$\max \{c^T x \mid x \in P_I\} \geq \max \{c^T x \mid x \in Q\} \forall c \in \mathbb{R}^n$$

Consider c with $\max \{c^T x \mid x \in P_I\} < \infty$, $x^* \in P_I \cap \mathbb{Z}^n$ maximizes $c^T x$

$$\Rightarrow \max \{c^T x \mid Ax \leq b\} < \infty$$

$$\Rightarrow \min \{y^T \cdot b \mid y^T A = c^T, y \geq 0\} \text{ feasible}$$

$$\Rightarrow c \in C$$

$$\text{Let } \bar{K} := \{z \in L \mid A \cdot (x^* + z) \leq b\}$$

Since x^* is optimal $\Rightarrow c^T \cdot z \leq 0 \forall z \in \bar{K} \Rightarrow c \in C_{\bar{K}}$

$$\Rightarrow c = \sum_{y \in G(\bar{K})} \lambda_y \cdot y \quad \lambda_y \geq 0$$

Claim: $\delta_y := \max \{y^T \cdot x \mid x \in P_I\} = y^T x^* \forall y \in G(\bar{K})$

Proof: Otherwise $\exists z \in \bar{K}$ with $y^T \cdot z > 0 \not\Leftarrow y \in C_{\bar{K}}$.

\Rightarrow For $x \in Q$:

$$c^T x = \sum_{y \in G(\bar{K})} (\lambda_y \cdot y^T) \cdot x$$

$$\leq \sum_{\gamma \in G(\bar{K})} \lambda_{\gamma} \cdot \delta_{\gamma}$$

$$= \sum_{\gamma \in G(\bar{K})} (\lambda_{\gamma} \cdot \gamma^T) x^* = c^T \cdot x^* \quad \square$$

Total dual integrality

Def: A polyhedron P is integral if $P = P_{\mathbb{I}}$.

Some definitions and notions on polyhedra

Let $P = \{x \mid Ax \leq b\}$ polyhedron

If $c \neq 0$, $c^T x \leq \delta \quad \forall x \in P$ ($c^T x \leq \delta$ is a "valid inequality" for P)

and $P \cap \{x \mid c^T x = \delta\} \neq \emptyset$ then

$\{x \mid c^T x = \delta\}$ is called a supporting hyperplane of P .

A face of P is P itself or the intersection of P with a supporting hyperplane.

A face of a polyhedron is itself a polyhedron.

A facet of P is a maximal face distinct from P .

Theorem: Let P be a rational polyhedron. The following statements are equivalent

a) P is integral

b) Each face of P contains integral vectors.

c) Each minimal face of P contains integral vectors.

- d) Each supporting hyperplane contains integral vectors
- e) Each rational support. hyperpl. " " vectors
- f) $\max \{c^T x \mid x \in P\} = \max \{c^T x \mid x \in P \cap \mathbb{Z}^n\} \forall c \in \mathbb{R}^n$
- g) $\max \{c^T x \mid x \in P\} \in \mathbb{Z} \cup \{\infty\} \forall c \in \mathbb{Z}^n$