

# Total dual integrality

P polyhedron

P integral if  $P = P_I$

Theorem: The following are equivalent:

- P integral
- Each face of P contains integral vectors
- Each min. face " " " "
- Each supp. hyperpl. " " " "
- Each rational supp. hyperpl. " " " "
- $\max \{c^T x \mid x \in P\} = \max \{c^T x \mid x \in P \cap \mathbb{Z}^n\} \quad \forall c \in \mathbb{R}^n$
- $\max \{c^T x \mid x \in P\} \in \mathbb{Z} \cup \{\infty\} \quad \forall c \in \mathbb{Z}^n$

Def.:  $Ax \leq b$  is called totally dual integral (TDI) if for each  $c \in \mathbb{Z}^n$  with  $\max \{c^T x \mid Ax \leq b\} < \infty$ :  
 $\max \{c^T x \mid Ax \leq b\} = \min \{y^T b \mid y^T A = c^T, y \geq 0, y \in \mathbb{Z}^m\}$

Corollary: Let  $A \in \mathbb{Q}^{m \times n}$ ,  $b \in \mathbb{Z}^m$  and  $Ax \leq b$  TDI. Then  $\{x \mid Ax \leq b\}$  is integral.

Proof: Follows from the definition of TDI and "g)  $\Rightarrow$  a)" of the above theorem.  $\square$

Application / Example:

Given: Directed graph  $D = (V, A)$ ,  $s, t \in V$

$\mathcal{P} = \{s-t\text{-paths in } D\}$ ,  $c: A \rightarrow \mathbb{R}_+$

Problem: Assign weights  $y_a \geq 0, a \in A$ , to the arcs such that the weight of any s-t-path is at least 1. Objective: minimize  $\sum_{a \in A} c(a) \cdot y_a$

LP Formulation:

$$\min \sum_{a \in A} c(a) \cdot y_a$$

$$\text{s.t. } \sum_{a \in P} \gamma_a \geq 1 \quad \forall P \in \mathcal{P}$$

$$\gamma_a \geq 0$$

Claim: This polyhedron  $Q = \{ \gamma \mid \sum_{a \in P} \gamma_a \geq 1 \quad \forall P \in \mathcal{P}, \gamma \geq 0 \}$  is integral because it is TDI.

$$\max \sum_{P \in \mathcal{P}} 1 \cdot x_P$$

$$\text{s.t. } \sum_{P: a \in P} x_P \leq c(a) \quad \forall a \in A$$

$$x_P \geq 0 \quad \forall P \in \mathcal{P}$$

This is a formulation of the max s-t-flow problem. If the "capacities"  $c(a)$  are integral  $\forall a \in A$ , then there exists an integral maxflow.

The corollary above thus implies that  $Q$  is integral.

Notice: TDI-ness is not a property of polyhedra but a property of linear inequality systems. There can be two different descriptions of a particular polyhedron by inequality systems such that one is TDI and the other one is not TDI.

Proposition: If  $Ax \leq b$  is TDI and  $a^T x \leq \beta$  for all  $x$  with  $Ax \leq b$ , then  $Ax \leq b, a^T x \leq \beta$  is also TDI.

Proof: For  $c \in \mathbb{Z}^n$

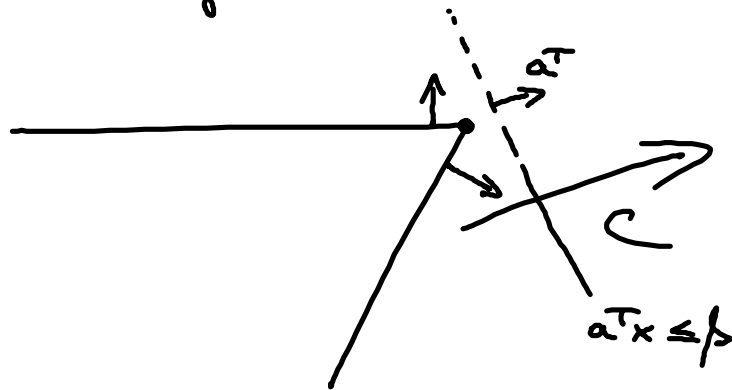
$$\min \{ y^T b \mid y^T A = c^T, y \geq 0 \} = \max \{ c^T x \mid Ax \leq b \}$$

(\*)

$$= \max \{ c^T x \mid Ax \leq b, a^T x \leq \beta \}$$

$$= \min \{ \gamma^T b + \gamma \cdot \beta \mid \gamma^T A + \gamma \cdot a^T = c^T, \gamma, \gamma \geq 0 \}$$

Since  $Ax \leq b$  is TPI,  $(*)$  has an integral opt. sol.  $\gamma$ . Then,  $(\gamma, 0)$  is an integral opt. sol. to  $(**)$ . □

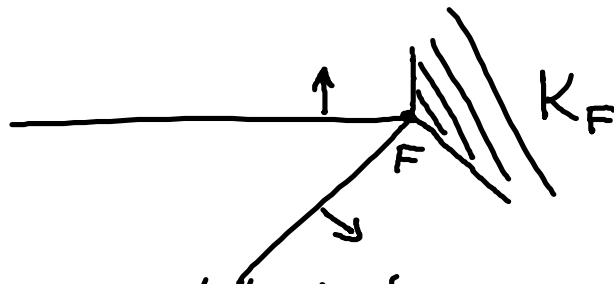


Theorem: For each rational polyhedron  $P$  there exists  $Ax \leq b$  TPI with  $A$  integral and  $P = \{ x \mid Ax \leq b \}$ . In particular  $b \in \mathbb{Z}^m \Leftrightarrow P$  integral.

Sketch of the proof:

Let  $P = \{ x \mid Cx \leq d \}$ ,  $C \in \mathbb{Q}^{m \times n}$ ,  $d \in \mathbb{Q}^m$  and  $F$  minimal face of  $P$ ,  $F = \{ x \mid C'x = d' \}$  with  $C'x \leq d'$  a subsystem of  $Cx \leq d$ .

$$K_F := \{ c \mid c^T z = \max \{ c^T x \mid x \in P \} \forall z \in F \}$$



The  $K_F$  is a polyhedral cone generated by the rows of  $C'$ . (omit details)

Let  $a_1, \dots, a_t$  be a Hilbert basis of  $K_F$  and

Let  $\max \{a_i^T x \mid x \in P\} = b_i$

Consider the resulting system  $Ax \leq b$  (for all  $F$ ).

(If  $P$  integral, then  $b$  integral)

$$\rightarrow P = \{x \mid Ax \leq b\}$$

"1" by Def. of r.h.s.  $b_i$ .

"2" Positive multiples of the rows of  $C$  must occur as rows of  $A$  since the rows of  $C^i$  generate the polyh. cone  $K_F$ .

It remains to prove that  $Ax \leq b$  is T.D.I.

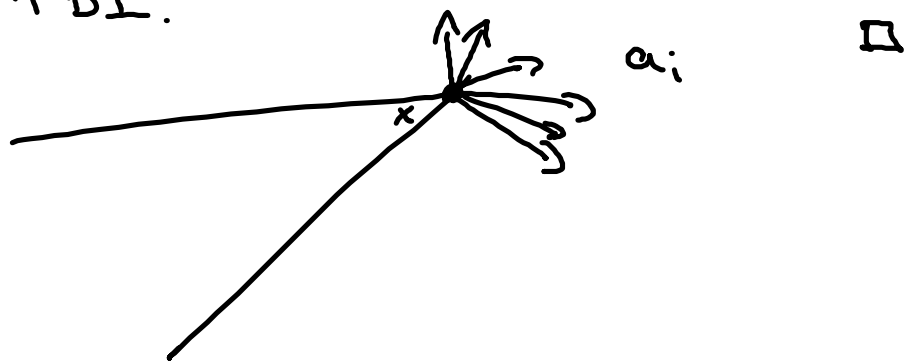
Let  $c \in \mathbb{Z}^n$  and  $F$  minimal face of  $P$  with  $\max \{c^T x \mid x \in P\} = c^T z \quad \forall z \in F$

$$\rightarrow c \in K_F, \quad c = \sum_{i=1}^t \lambda_i a_i, \quad \lambda_i \in \mathbb{Z}_+$$

$$\Rightarrow c^T = \lambda^T A \quad \text{for some } \lambda \in \mathbb{Z}_+^{m'}$$

$$\Rightarrow \lambda^T \cdot b = \lambda^T \cdot (Ax) = (\lambda^T A) \cdot x = c^T \cdot x \quad \forall x \in F$$

$$\Rightarrow Ax \leq b \text{ is T.D.I.}$$



Theorem: Let  $Ax \leq b, a^T x = \beta$  T.D.I. and  $a^T$  integral  $\rightarrow Ax \leq b, a^T x = \beta$  T.D.I.