Technische Universität Berlin
Fakultät II – Institut für Mathematik

Berlin Mathematical School
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Topology

Dr. Carsten Schultz

Axel Werner

Problem Set 1

Deadline: 29 Oct 2008

Exercise 1. **4 points**

Show that for every $x \in \mathbb{R}^n$ and $\varepsilon > 0$ the open ball $B_\varepsilon(x) \subset \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$.

Exercise 2. **4 points**

Let X be the subspace of $\mathbb{R}$, defined by

$$X \ := \ \{0\} \cup \left\{ \frac{1}{n} \ \middle| \ n \in \mathbb{N} \setminus \{0\} \right\}.$$

Show that a bijection $f : X \to X$ is a homeomorphism if and only if $f(0) = 0$.

Exercise 3. **4 points**

Let X and Y be the following subsets of $S^3 \subset \mathbb{R}^4$:

$$\begin{aligned} X &:= \left\{ (x_1, x_2, x_3, x_4) \in S^3 \mid x_1^2 + x_2^2 \geq x_3^2 + x_4^2 \right\} \\ Y &:= \left\{ (x_1, x_2, x_3, x_4) \in S^3 \mid x_1^2 + x_2^2 \leq x_3^2 + x_4^2 \right\} \end{aligned}$$

Obviously, $S^3 = X \cup Y$.

(a) Show that

- $X \cap Y \approx S^1 \times S^1$,
- $X \approx S^1 \times D^2$,
- $Y \approx D^2 \times S^1$.

(b) Sketch the situation in “half” dimension, that is, with $\mathbb{R}^4$, S^3, S^1 and D^2 replaced by $\mathbb{R}^2$, S^1, S^0 and D^1, respectively.

Hint: It might help to solve part (b) first. Of the three homeomorphisms the first one is easiest to describe.

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Exercise 4. **4 points**

Let X be a topological space and $A, B \subseteq X$, $x \in X$.

(a) Show that the following statements are equivalent:

(i) $A = \overline{B}$.

(ii) A is the smallest closed subset containing B; that is, A is closed, $B \subseteq A$ and for all closed sets C with $C \supseteq B$ we have $A \subseteq C$.

(b) Show that the following statements are equivalent:

(i) $x \in \overline{B}$.

(ii) For every neighbourhood U of x we have $U \cap B \neq \emptyset$.

(c) Formulate analogous characterisations for $A = \operatorname{int} B$ and $x \in \operatorname{int} B$. (You don't have to prove them.)

Exercise 5. **(Tutorial)**

Let $X := [0, 1)$ and

$$\begin{aligned} d : X \times X &\to \mathbb{R} \\ (x, y) &\mapsto \min\{|x - y|, x + |1 - y|, |x - 1| + y\} \end{aligned}$$

(a) Show that d defines a metric on X.

(b) Show that X with the metric d is homeomorphic to S^1.

Exercise 6. **(Tutorial)**

Let X be a T_2-space. Show that for every topological space Z, every dense set $D \subseteq Z$ and every two continuous mappings $f, g : Z \to X$ such that $f|_D = g|_D$ we have $f = g$.

Exercise 7. **(Tutorial)**

Let $e_{n+1} := (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Show that $\mathbb{R}^n \approx S^n \setminus \{e_{n+1}\}$.