

Topology

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Problem Set 1

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Exercise 1.

4 points

Show that for every $x \in \mathbb{R}^n$ and $\varepsilon > 0$ the open ball $B_\varepsilon(x) \subset \mathbb{R}^n$ is homeomorphic to \mathbb{R}^n .

Exercise 2.

4 points

Let X be the subspace of \mathbb{R} , defined by

$$X := \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{N} \setminus \{0\} \right\}.$$

Show that a bijection $f : X \rightarrow X$ is a homeomorphism if and only if $f(0) = 0$.

Exercise 3.

4 points

Let X and Y be the following subsets of $S^3 \subset \mathbb{R}^4$:

$$\begin{aligned} X &:= \{(x_1, x_2, x_3, x_4) \in S^3 \mid x_1^2 + x_2^2 \geq x_3^2 + x_4^2\} \\ Y &:= \{(x_1, x_2, x_3, x_4) \in S^3 \mid x_1^2 + x_2^2 \leq x_3^2 + x_4^2\} \end{aligned}$$

Obviously, $S^3 = X \cup Y$.

(a) Show that

- $X \cap Y \approx S^1 \times S^1$,
- $X \approx S^1 \times D^2$,
- $Y \approx D^2 \times S^1$.

(b) Sketch the situation in “half” dimension, that is, with \mathbb{R}^4 , S^3 , S^1 and D^2 replaced by \mathbb{R}^2 , S^1 , S^0 and D^1 , respectively.

Hint: It might help to solve part (b) first. Of the three homeomorphisms the first one is easiest to describe.

Exercise 4.**4 points**

Let X be a topological space and $A, B \subseteq X$, $x \in X$.

(a) Show that the following statements are equivalent:

(i) $A = \overline{B}$.

(ii) A is the smallest closed subset containing B ; that is, A is closed, $B \subseteq A$ and for all closed sets C with $C \supseteq B$ we have $A \subseteq C$.

(b) Show that the following statements are equivalent:

(i) $x \in \overline{B}$.

(ii) For every neighbourhood U of x we have $U \cap B \neq \emptyset$.

(c) Formulate analogous characterisations for $A = \text{int } B$ and $x \in \text{int } B$. (You don't have to prove them.)

Exercise 5.**(Tutorial)**

Let $X := [0, 1)$ and

$$d : X \times X \rightarrow \mathbb{R} \\ (x, y) \mapsto \min \{|x - y|, x + |1 - y|, |x - 1| + y\}$$

(a) Show that d defines a metric on X .

(b) Show that X with the metric d is homeomorphic to S^1 .

Exercise 6.**(Tutorial)**

Let X be a T_2 -space. Show that for every topological space Z , every dense set $D \subseteq Z$ and every two continuous mappings $f, g : Z \rightarrow X$ such that $f|_D = g|_D$ we have $f = g$.

Exercise 7.**(Tutorial)**

Let $e_{n+1} := (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$. Show that $\mathbb{R}^n \approx S^n \setminus \{e_{n+1}\}$.