

## Topology

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### Problem Set 2

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#### Exercise 8.

4 points

Let  $X$  be a topological space. Show that the following statements are equivalent:

- (a)  $X$  is a  $T_2$ -space.
- (b) The diagonal  $\Delta := \{(x, x) \in X \times X\}$  is closed in  $X \times X$ .
- (c) For every space  $Z$ , every dense subset  $D \subseteq Z$  and every two continuous maps  $f, g : Z \rightarrow X$  such that  $f|_D = g|_D$  we have  $f = g$ .

#### Exercise 9.

4 points

Show:

- (a)  $\mathbb{R}$  is not homeomorphic to  $\mathbb{R}^2$ .
- (b)  $S^1$  is not homeomorphic to  $S^2$ .

*Hint:* Exercise 7.

**Definition.** Let  $X$  be a topological space,  $M$  a set and  $f : X \rightarrow M$  a function.  $f$  is called *locally constant*, if every  $x \in X$  has a neighbourhood  $U$  such that  $f|_U$  is constant.

#### Exercise 10.

4 points

Consider the following subspace of  $\mathbb{R}^2$ :

$$X := \left( \left\{ \frac{1}{n} \mid n \in \mathbb{N} \setminus \{0\} \right\} \times I \right) \cup \{(0, 0), (0, 1)\}$$

- (a) Determine the connected components of  $X$ .
- (b) Show that for every locally constant function  $f : X \rightarrow M$  to some set  $M$  we have  $f(0, 0) = f(0, 1)$ .

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**Exercise 11.****4 points**

Let  $n \geq 2$  and  $A$  a countable subset of  $\mathbb{R}^n$ . Show that  $\mathbb{R}^n \setminus A$  is path connected.

**Exercise 12.****(Tutorial)**

Let  $X$  be a topological space,  $M$  a set and  $f : X \rightarrow M$  a locally constant mapping. Show:

- (a) Given any  $y \in M$ , the set  $\{x \in X \mid f(x) = y\}$  is open and closed.
- (b) If  $X$  is connected then  $f$  is constant.

**Exercise 13.****(Tutorial)**

Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and define

$$\begin{aligned} d_{X \times Y} : (X \times Y) \times (X \times Y) &\rightarrow \mathbb{R} \\ ((x, y), (x', y')) &\mapsto d_X(x, x') + d_Y(y, y'). \end{aligned}$$

Show:

- (a)  $(X \times Y, d_{X \times Y})$  is a metric space.
- (b) The induced topology on  $X \times Y$  by  $d_{X \times Y}$  is the product topology of the topologies on  $X$  and  $Y$ , induced by  $d_X$  and  $d_Y$ , respectively.

**Exercise 14.****(Tutorial)**

Let  $X$  and  $Y$  be topological spaces and  $A \subset X$ ,  $B \subset Y$  with the subspace topologies. Show that the product topology on  $A \times B$  coincides with the subspace topology of  $A \times B$  as a subset of  $X \times Y$ .

**Exercise 15.****(Tutorial)**

Let  $X$  be a topological space and  $A, B \subseteq X$  closed subsets such that  $X = A \cup B$ . Then a function  $f : X \rightarrow Y$  is continuous if and only if both  $f|_A$  and  $f|_B$  are continuous.

