Technische Universität Berlin
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Topology

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Problem Set 4

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Definition. Let X and Y be topological spaces and $f : X \to Y$ a function. f is *sequentially continuous* if for every $x \in X$ and every sequence $(a_n)_{n\in\mathbb{N}}$ that converges to x, the sequence $(f(a_n))_{n\in\mathbb{N}}$ converges to $f(x)$.

Exercise 23. **6 points**

Show:

(a) Every continuous function is sequentially continuous.

(b) If X and Y are spaces, where X is first countable, and $f : X \to Y$ is sequentially continuous, then f is continuous.

(c) There are spaces X, Y and a sequentially continuous function $f : X \to Y$ which is not continuous.

Hint: You can use that there is a space Z with a subset A and a point $x \in \overline{A}$ such that no sequence in A converges against x.

Exercise 24. **6 points**

Let $\mathcal{F}$ be an ultra-filter on $\mathbb{N}$ which refines the filter induced by the sequence $(n)_{n\in\mathbb{N}}$. We fix this ultra-filter for the rest of this exercise.

(a) Show that for every bounded sequence $a : \mathbb{N} \to \mathbb{R}$ there is a unique $x \in \mathbb{R}$ such that $a_*(\mathcal{F}) \to x$. Let us denote this number x by $\textsc{lim}\, a$.

(b) Show that $\textsc{lim}\, a$ is an accumulation point of the sequence a.

(c) Let a, b be bounded sequences in $\mathbb{R}$. Show that $\textsc{lim}(a \cdot b) = (\textsc{lim}\, a) \cdot (\textsc{lim}\, b)$ and $\textsc{lim}(a + b) = (\textsc{lim}\, a) + (\textsc{lim}\, b)$.

Hint: Multiplication and addition are continuous functions $\mathbb{R}^2 \to \mathbb{R}$.

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Exercise 25. **4 points**

Let $G = (V, E)$ be an infinite graph with vertices in V and edges in $E \subseteq V \times V$. Suppose C is a finite set, whose elements we call *colours*. A *(vertex-)colouring* of G (with colours in C) is a function $V \to C$ and a colouring c is called *admissible* if any two connected vertices have different colours, that is, $c(e) \neq c(e')$ for all edges $(e, e') \in E$.

Show that G has an admissible colouring if every finite subgraph of G has an admissible colouring.

Hint: Use Tychonoff's Theorem to choose a topology on the set of all colourings, which is compact and has the property that for every edge the set of colourings that are "admissible at this edge" is closed. Then use Lemma 4.24.

Exercise 26. **(Tutorial)**

Let S be a set, a a sequence in S, and $\mathcal{F}_a$ the induced filter on S. Show that $\mathcal{F}_a$ is an ultra-filter if and only if the sequence a is eventually constant.

Exercise 27. **(Tutorial)**

Let X be a topological space and $\mathcal{F}$ an ultra-filter on X. Show that if $n \in \mathbb{N}$ and $A_i \subseteq X$ for all $i \in \{1, \ldots, n\}$ such that $\bigcup_{i=1}^{n} A_i = X$, then there exists an $i \in \{1, \ldots, n\}$ with $A_i \in \mathcal{F}$.

Exercise 28. **(Tutorial)**

Let $\mathcal{F}$ be an ultra-filter on $\mathbb{N}$. Show that exactly one of the following statements is true.

1. There is an $n \in \mathbb{N}$ such that $\mathcal{F} = \{A \subset \mathbb{N} \mid n \in A\}$.
2. $\mathcal{F}$ is a refinement of the filter induced by the sequence $(n)_{n \in \mathbb{N}}$.

Exercise 29. **(Tutorial)**

Consider the function

$$i : \mathbb{N} \to \{0, 1\}^{\mathcal{P}(\mathbb{N})}$$

$$i(n)(A) := \chi_A(n) := \begin{cases} 1, & n \in A, \\ 0, & n \notin A. \end{cases}$$

Show the following statements.

(a) The map i is an embedding and its image is not closed.

(b) Let $f \in \{0, 1\}^{\mathcal{P}(\mathbb{N})}$. Then $f \in \overline{\operatorname{Im} i}$ if and only if $\mathcal{F} := \{A \subset \mathbb{N} \mid f(A) = 1\}$ is an ultra-filter on $\mathbb{N}$. In this case $i_*(\mathcal{F}) \to f$.

(c) The sequence i has an accumulation point but no convergent subsequence.