

## Topology

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### Problem Set 5

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#### Exercise 30.

6 points

(a) Let  $(X, d)$  be a metric space with the induced topology. Show that

$$d'(x, y) := \min \{d(x, y), 1\}$$

defines a metric on  $X$  that induces the same topology as  $d$ .

(b) For every  $n \in \mathbb{N}$  let  $(X_n, d_n)$  be a metric space with  $d_n$  bounded by 1. Show that

$$d(x, y) := \sum_{n=0}^{\infty} 2^{-n} d(x_n, y_n)$$

defines a metric on the product  $\prod_{n \in \mathbb{N}} X_n$ .

(c) Show that countable products of metrisable spaces are metrisable.

#### Exercise 31.

4 points

Let  $X := (\mathbb{R}, \mathcal{T})$  with

$$\mathcal{T} := \{O \setminus A \mid O, A \subseteq \mathbb{R}, O \text{ open w.r.t. the Euclidean topology, } A \text{ countable}\}.$$

Show:

(a)  $X$  is a topological space.

(b)  $X$  is  $T_2$ .

(c)  $X$  is not regular.

**Exercise 32.****6 points**

Let  $X$  be a normal space and  $A$  a closed subset of  $X$ . Show:

- (a) If  $f : A \rightarrow \mathbb{R}^n$  is continuous and bounded, then there exists a continuous extension of  $f$  to  $X$ .
- (b) If  $f : A \rightarrow S^n$  is continuous, then there is a neighbourhood  $U$  of  $A$  such that there exists a continuous extension  $F : U \rightarrow S^n$  of  $f$ .  
*Hint:* A suitable continuous function  $\mathbb{R}^{n+1} \setminus \{0\} \rightarrow S^n$  might help...
- (c) Give an example of a space  $X$ , a closed subset  $A \subset X$  and a continuous function  $f : A \rightarrow S^0$  that does not have a continuous extension  $F : X \rightarrow S^0$ .

**Exercise 33.****(Tutorial)**

Let  $X$  be a topological space and  $D$  a dense subset of  $X$ . Show that if there exists a closed discrete subspace  $S$  of  $X$  with  $|S| \geq |\mathcal{P}(D)|$  then  $X$  is not normal.

**Exercise 34.****(Tutorial)**

Define on  $\mathbb{R}$  the topology  $\mathcal{T}$  which has the basis  $\{[a, b) \mid a, b \in \mathbb{R}\}$ .

- (a) Show that  $(\mathbb{R}, \mathcal{T})$  is normal.
- (b) Show that  $X := (\mathbb{R}, \mathcal{T}) \times (\mathbb{R}, \mathcal{T})$  is not normal.

*Hint:* Use Exercise 33 with  $D = \mathbb{Q} \times \mathbb{Q}$  and  $S = \{(x, -x) \mid x \in \mathbb{R}\}$ .

*Recall:* A space  $X$  is  $T_0$ , if for every two distinct points in  $X$  there exists an open set which contains one of the points, but not the other.

**Exercise 35.****(Tutorial)**

Show:

- (a) Every finite  $T_1$  space has the discrete topology.
- (b) There are finite  $T_0$  spaces that are not discrete.

**Exercise 36.****(Tutorial)**

Let  $X$  be a  $T_2$  space and  $x_1, \dots, x_n \in X$  distinct points. Show that there are disjoint neighbourhoods  $U_1, \dots, U_n$  with  $x_i \in U_i$  for  $1 \leq i \leq n$ .