Technische Universität Berlin
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Wintersemester 2008/2009

Topology

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Problem Set 7

Deadline: 10 Dec 2008

Exercise 42. **4 points**

Let $e : X \to K$ be a compactification of a locally compact space. Prove the following statements:

(a) The subset $e[X]$ of K is open.

(b) There is a morphism from e to the one-point-compactification of X.

Hint: There seem to be essentially two ways to do this. The more direct one is to prove (a) first and by elementary methods and to use it to prove the continuity of a morphism that one defines to prove (b).
The possibly more elegant solution uses the Stone-Čech-compactification of X, Proposition 6.23, Exercise 39, and Proposition 2.32 to obtain the morphism that proves (b) and then obtains (a) as a consequence.

Exercise 43. **4 points**

Show that the following spaces are homeomorphic:

(i) $S^1 \times S^1$,

(ii) $\mathbb{R}^2/\sim$ with $(x, y) \sim (x', y') :\Leftrightarrow (x - x', y - y') \in \mathbb{Z} \times \mathbb{Z}$,

(iii) $(I \times I)/\sim$, where $\sim$ is induced by $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$ for all $x, y \in I$.

Exercise 44. **4 points**

Let A be a closed subset of a compact space X. Show that X/A is compact.

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❄ Exercise 45. **4 extra points**

Let $\mathsf{C}\mathbb{N}$ the cone over $\mathbb{N}$ and

$$X := \left\{ (n(1-\lambda), \lambda) \in \mathbb{R}^2 \mid \lambda \in [0,1], n \in \mathbb{N} \right\}.$$

Show:

(a) There exists a continuous bijection $\mathsf{C}\mathbb{N} \to X$.

(b) $\mathsf{C}\mathbb{N}$ is not homeomorphic to X.
Hint: Is $\mathsf{C}\mathbb{N}$ first countable?

Exercise 46. **(Tutorial)**

Let $\sim$ be the equivalence relation on $X := \mathbb{R} \times \{0,1\}$, defined by

$$(x,n) \sim (y,m) \;\Leftrightarrow\; (x = y) \wedge ((x \neq 0) \vee (n = m)).$$

Show:

(a) $X/\sim$ is not T_2.

(b) Every point in $X/\sim$ has a neighbourhood homeomorphic to $\mathbb{R}$.

Exercise 47. **(Tutorial)**

What is $\mathbb{R}/\sim$ with $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$?

Exercise 48. **(Tutorial)**

Let $f : X \to Y$ be a function between compact spaces such that the graph of f is a closed subset of $X \times Y$.

(a) Show that f is continuous.

(b) Is compactness of X really needed? What about the compactness of Y?

Exercise 49. **(Tutorial)**

Let $e : X \to K$ be a compactification such that $e[X]$ is open in K. Show that X is locally compact.