

Topology

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Problem Set 8

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Exercise 50.

4 points

Let X be a contractible space and Y a *retract* of X , that is, a subspace of X such that there exists a map $r : X \rightarrow Y$ with $r|_Y = \text{id}_Y$. Prove that Y is contractible.

Exercise 51.

4 points

Let X be a space. Show that CX is contractible.

Exercise 52.

4 points

Let $p_n : S^n \rightarrow \mathbb{R}P^n$ the quotient map from the definition of the projective space $\mathbb{R}P^n$. Show that $\mathbb{R}P^0$ is a space containing only one point, and $\mathbb{R}P^{n+1} \approx \mathbb{R}P^n \cup_{p_n} D^{n+1}$.

Exercise 53.

4 points

Consider $S^1 \times S^1 \subset S^1 \times D^2$ and the mappings

$$\begin{aligned} i : S^1 \times S^1 &\rightarrow S^1 \times D^2 & t : S^1 \times S^1 &\rightarrow S^1 \times D^2 \\ (x, y) &\mapsto (x, y), & (x, y) &\mapsto (y, x). \end{aligned}$$

Show:

(a) $S^1 \times D^2 \cup_i S^1 \times D^2 \approx S^1 \times S^2$,

(b) $S^1 \times D^2 \cup_t S^1 \times D^2 \approx S^3$.

Hint: Exercise 3.

Exercise 54.**(Tutorial)**

Let X be the subspace of \mathbb{R}^2 defined by

$$X := \left(\left(\left\{ \frac{1}{n} \mid n \in \mathbb{N} \setminus \{0\} \right\} \cup \{0\} \right) \times I \right) \cup (I \times \{1\})$$

Let $x_0 := (0, 0) \in X$. Show that $\text{id}_X \simeq c_{x_0}$ but $\text{id}_X \not\approx c_{x_0} \text{ rel } \{x_0\}$.

Exercise 55.**(Tutorial)**

For $k \geq -1$, a space X is called k -connected, if every continuous map $S^\ell \rightarrow X$, $-1 \leq \ell \leq k$, can be extended to a continuous map $D^{\ell+1} \rightarrow X$.

- (a) Show that if $X \neq \emptyset$ is contractible then X is k -connected for all k .
- (b) For which k do you think S^n is k -connected?

Exercise 56.**(Tutorial)**

Let X, Y be spaces, $A \subset X$, $A \neq \emptyset$, $p : X \rightarrow X/A$ the quotient map, $\{x_0\} = p[A]$ and $y_0 \in Y$. Show that there is a bijection between the sets of equivalence classes

$$\{f : X \rightarrow Y \mid f \text{ continuous, } f[A] = \{y_0\}\} /_{f \sim g: \Leftrightarrow f \simeq g \text{ rel } A}$$

and

$$\{f : X/A \rightarrow Y \mid f \text{ continuous, } f(x_0) = y_0\} /_{f \sim g: \Leftrightarrow f \simeq g \text{ rel } \{x_0\}}.$$

Remark: Important special cases are $X = D^n$, $A = S^{n-1}$ and $X = I$, $A = \{0, 1\}$.

