Average volume, curvatures, and Euler characteristic of random real algebraic varieties

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Abstract

We determine the expected curvature polynomial of random real projective varieties given as the zero set of independent random polynomials with Gaussian distribution, whose distribution is invariant under the action of the orthogonal group. In particular, the expected Euler characteristic of such random real projective varieties is found. This considerably extends previously known results on the number of roots, the volume, and the Euler characteristic of the solution set of random polynomial equations.

Key words. Random polynomials, real zeros of random polynomial equations, Euler characteristic, volume of tubes, curvature polynomial, kinematic formula, orthogonal invariance

AMS subject classifications. 60D05, 14P25, 53C65, 60G60, 60G15

1 Introduction

1.1 Real zeros of random polynomials

The study of the real zeros of random polyomials started with a paper by Bloch and Pólya [6] who investigated polynomials in one variable with independent random coefficients from the set $\{-1, 0, 1\}$. The question was further investigated by Littlewood and Offord [25, 26] who estimated the average number of roots with respect to different probability distributions of the coefficients. The first asymptotically sharp estimate on the average number of real roots was obtained Kac [21]. He showed that the expected number of real roots of a random degree d polynomial is asymptotically $\frac{2}{\pi} \ln d$ if the coefficients are independent and standard normal distributed. Erdös and Offord [14] obtained similar results for random coefficients in $\{-1, 1\}$.

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Maslova [28, 29] proved that Kac's asymptotic result in fact holds for large classes of distributions. For more details we refer to the textbook by Bharucha-Reid and Sambandham [5]. Edelman and Kostlan [13] give a very nice account of this and related work.

The paper by Shub and Smale [36] was a breakthrough for the study of real roots of random systems of polynomial equations. The key point in that work is an assumption on the underlying probability measure that is very natural from a geometric point of view, namely invariance under the action of the orthogonal group. By sharp contrast with Kac's result [21], the expected number of real roots of a random degree d polynomial $f = \sum_{\alpha=0}^{d} f_\alpha X^\alpha$ turns out to be exactly $\sqrt{d}$ if the coefficients f_α are independent centered Gaussian random variables with variance $\binom{d}{\alpha}$. To understand the invariance, the polynomials should be interpreted as bivariate forms (having roots in $\mathbb{P}^1$). The resulting probability distribution on bivariate forms is invariant under the action of the orthogonal group $O(2)$.

More generally, let $H_{d,n}$ denote the vector space of homogeneous real polynomials of degree d in the variables $X_0, \dots, X_n$. Let $f = \sum_\alpha f_\alpha X_0^{\alpha_0} \cdots X_n^{\alpha_n} \in H_{d,n}$ be homogeneous of degree d such that the coefficients f_α are independent centered Gaussian random variables with variance $\binom{d}{\alpha} := \frac{d!}{\alpha_0! \cdots a_n!}$. The induced probability distribution on the space of forms of degree d can be shown to be invariant under the natural action of the orthogonal group $O(n+1)$. We will say that f is a *Kostlan distributed* random polynomial. Consider a system $f_1(x) = 0, \dots, f_n(x) = 0$ of Kostlan distributed random polynomials. Shub and Smale [36] showed that its expected number of roots equals $\sqrt{d_1 \cdots d_n}$, i.e., the square root of the product of the degrees of the polynomials. This result was also found by Kostlan [23] in the case where all polynomials have the same degree. In fact, Shub and Smale's result was a byproduct of a probabilistic analysis of nonlinear condition numbers that control the cost of a projective homotopy method to solve systems of polynomial equations, see also [7]. For a recent extension of such probabilistic analysis to the framework of "smoothed analysis" we refer to [10, 11]. We remark that the above choice of invariant probability measure seems also natural from the point of view of physics [8].

The results by Kostlan [23] and Shub and Smale [36] on the expected number of real roots in the setting of an invariant probability measure have been extended to multihomogeneous systems by McLennan [30] and, partially, to sparse systems by Rojas [34] and Malajovich and Rojas [27]. The work of Kostlan [24] contains a classification of Gaussian invariant random polynomials along with further results. Recently, Azaïs and Wschebor [3] gave a new proof of the Shub-Smale theorem based on the Rice formula from the theory of random fields. Wschebor [44], for the first time, analyzed the variance of the number of real roots.

1.2 Underdetermined random polynomial systems

Considerably less is known for the underdetermined case $f_1(x) = 0, \ldots, f_s(x) = 0$ ($s < n$) where the set of solutions is a real algebraic variety of positive dimension. As a measure of its size different choices come to mind. One possible choice is the volume, which is finite when the solution set is interpreted in projective space. Another generalization of cardinality to higher dimensional solutions sets is the Euler characteristic. (This generalization is not only natural from the topological, but also from the computational complexity point of view, as shown in [9].) Both of these measures have been considered already.

Kostlan [23] showed that for a system of Kostlan distributed f_i, the volume of the projective solution set has the expectation

$$\mathbb{E}\,\mathrm{vol}(\mathcal{Z}(f_1, \ldots, f_s)) = d^{s/2}\,\mathrm{vol}(\mathbb{P}^{n-s}) \tag{1}$$

in the case $d_1 = \ldots = d_s = d$. (Unfortunately, Kostlan never published the proof. A proof has been given in [33].)

Podkorytov [32] considered *any* centered Gaussian random polynomial f that is invariant under the action of the orthogonal group and determined the expected Euler characteristic of its zero set $\mathcal{Z}(f)$ in projective space $\mathbb{P}^n$. Podkorytov showed

$$\mathbb{E}\chi(\mathcal{Z}(f)) = I_n(\delta^{1/2})/I_n(1) \quad \text{for odd } n, \tag{2}$$

where δ is the *parameter* of f (see Definition 4.4) and $I_n(\delta) := \int_0^\delta (1 - x^2)^{\frac{n-1}{2}}\,dx$. If f is Kostlan distributed, then its parameter equals the degree d. Note that if n is even, $\mathcal{Z}(f)$ is almost surely a compact odd-dimensional manifold and therefore its Euler characteristic vanishes. Apparently, Podkorytov was not aware of Shub and Smale's work [36]. His proof is based on some tricky application of Morse theory. An application of Podkorytov's result can be found in [22].

1.3 Main results

The original motivation of the present work was to extend Podkorytov's result [32] for hypersurfaces to projective varieties of higher codimension. However, a direct use of Morse theory as in [32] does not seem feasible. It turned out to be essential to study the more general notions of curvature coefficients and curvature polynomial of a projective variety. Our main result (Theorem 1.1) considerably extends and unifies the previously known results on the number of roots, volume, and Euler characteristic. It determines the expectation of the curvature polynomial of a random projective variety under an invariant probability measure. Before stating our result, we need to explain the notion of curvature coefficients.

In a seminal work, Weyl [42] derived a formula for the volume of the tube $T(M, \alpha) := \{y \in S^n \mid \mathrm{dist}(y, M) \leq \alpha\}$ of radius α around an m-dimensional compact smooth submanifold M of the sphere S^n. Let $\mu := n - m$ denote the codimension of M. Weyl [42] proved that, for sufficiently small $\alpha > 0$, $\operatorname{vol}(T(M,\alpha))$ can be written as a linear combination

$$\operatorname{vol}(T(M,\alpha)) = \sum_{0\le e\le m,\, e \text{ even}} K_{\mu+e}(M) J_{\mu+e}(\alpha) \tag{3}$$

of the functions $J_k(\alpha) := \int_0^\alpha (\sin\rho)^{k-1}(\cos\rho)^{n-k} d\rho$. The coefficients $K_{\mu+e}(M)$ depend on the curvature of M in S^n and will be thus called the *curvature coefficients* of the submanifold M.

Let $\mathcal{O}_{n-1} := 2\pi^{n/2}/\Gamma(n/2)$ denote the $(n-1)$-dimensional volume of the unit sphere S^{n-1}. Following Nijenhuis [31] we rescale the curvature coefficients by

$$\mu_e(M) := \frac{1}{\mathcal{O}_{m-e}\mathcal{O}_{\mu+e-1}} K_{\mu+e}(M) \quad \text{ for } 0 \le e \le m,\ e \text{ even} \tag{4}$$

and define the *curvature polynomial* of the compact smooth submanifold M of S^n by setting

$$\mu(M;T) := \sum_{0\le e\le m,\, e \text{ even}} \mu_e(M) T^e. \tag{5}$$

For example, a subsphere S^m of S^n satisfies $\mu(S^m;T) = 1$. It is not hard to see that the constant term of the curvature polynomial describes the volume of M: we have $\mu_0(M) = \mathcal{O}_m^{-1}\operatorname{vol}(M)$. The generalized Gauss-Bonnet theorem of Allendoerfer and Weil [2] and Herglotz [18] implies that the Euler characteristic $\chi(M)$ of M can be retrieved by evaluating the curvature polynomial of M at 1: we have $\chi(M) = 2\,\mu(M;1)$ if m is even.

Using the canonical isometric 2-covering map $\pi\colon S^n \to \mathbb{P}^n$, we define the curvature polynomial of a compact smooth submanifold M of real projective space $\mathbb{P}^n$ by $\mu(M;T) := \mu(\pi^{-1}(M);T))$. It is then easy to see that $\mu_0(M) = \operatorname{vol}(\mathbb{P}^m)^{-1}\operatorname{vol}(M)$ and $\chi(M) = \mu(M;1)$.

Suppose now that $f = (f_1,\ldots,f_s) \in H_{d_1,n}\times\cdots\times H_{d_s,n}$ is a Gaussian random system of polynomials. It can be deduced from Sard's lemma that the hypersurfaces $\mathcal{Z}(f_i)$ intersect transversally almost surely, in which case the real projective zero $\mathcal{Z}(f_1,\ldots,f_s) \subseteq \mathbb{P}^n$ is a smooth projective variety of pure codimension s, or empty. To avoid this case distinction we define $\mu(\emptyset;T) := 0$.

We can now state our main result.

Theorem 1.1 *Suppose that $f_1 \in H_{d_1,n},\ldots,f_s \in H_{d_s,n}$ $(s \le n)$ are independent centered Gaussian random polynomials with a distribution that is invariant under the action of the orthogonal group $O(n+1)$. Let δ_σ denote the parameter of f_σ. Then the expected curvature polynomial of the projective variety $\mathcal{Z}(f_1,\ldots,f_s)$ in $\mathbb{P}^n$ is determined by*

$$\mathbb{E}\mu(\mathcal{Z}(f_1,\ldots,f_s);T) \;\equiv\; \prod_{\sigma=1}^{s} \frac{\delta_\sigma^{1/2}}{(1-(1-\delta_\sigma)T^2)^{1/2}} \bmod T^{n-s+1}.$$

In particular, $\mathbb{E}\mu_e(\mathcal{Z}(f_1,\ldots,f_s))$ depends only on $\delta_1,\ldots,\delta_s$ and e. When all parameters are equal to δ, we obtain

$$\mathbb{E}\mu(\mathcal{Z}(f_1,\ldots,f_s);T) \;=\; \delta^{s/2}\sum_{k=0}^{\lfloor\frac{n-s}{2}\rfloor} C_k^{(s)}(1-\delta)^k T^{2k},$$

where $C_0^{(s)}=1$ and $C_k^{(s)}=\frac{s(s+2)(s+4)\cdots(s+2k-2)}{k!\,2^k}$ for $k>0$.

For instance, the theorem implies $\mathbb{E}\,\mathrm{vol}(\mathcal{Z}(f_1,\ldots,f_s)) = (\delta_1\cdots\delta_s)^{1/2}\,\mathrm{vol}(\mathbb{P}^{n-s})$, which generalizes (1). In the case where all parameters equal δ and $n-s$ is even, the expected Euler characteristic satisfies

$$\begin{aligned}\mathbb{E}\chi(\mathcal{Z}(f_1,\ldots,f_s)) &= \delta^{s/2}\sum_{k=0}^{\frac{n-s}{2}} C_k^{(s)}(1-\delta)^k\\ &= (-1)^{\frac{n-s}{2}}\;C^{(s)}_{\frac{n-s}{2}}\;\delta^{\frac{n}{2}}+\mathcal{O}(\delta^{\frac{n}{2}-1})\quad(\delta\to\infty).\end{aligned}$$

1.4 Methods of proof

The proof of our main theorem is inspired by Azaïs and Wschebor's [3] new proof of the Shub-Smale theorem based on the Rice formula from the theory of random fields. Starting from Weyl's tube formula (3), we derive a version of a "Rice formula" for curvature coefficients (Theorem 5.2) and proceed by a careful probabilistic analysis of that formula, making heavily use of the orthogonal invariance.

In fact, for the proof of Theorem 1.1, it is sufficient to consider the case of one equation. This follows by employing a version of Chern's [12] kinematic formula of integral geometry for real projective space $\mathbb{P}^n$. According to Nijenhuis [31], the expected curvature polynomial of the intersection of a compact smooth submanifold M of $\mathbb{P}^n$ with a randomly moving compact submanifold N of $\mathbb{P}^n$ is a truncated product of the curvature polynomials of M and N, cf. Theorem 2.2. Actually this was shown by Chern and Nijenhuis for submanifolds of Euclidean space, however it also holds for projective space, cf. Santaló [35]. Details on this can be found in the monograph by Howard [20].

However, despite the considerable simplification of the arguments for the case of a hypersurface, we develop the proof via Weyl's tube formula and the Rice formula in full generality. Besides some intrinsic interest in obtaining a self-contained probabilistic proof, the main reason for doing so is that this avenue will also allow to treat higher moments (variance), as recently done so by Wschebor [44] for the case of a zero dimensional solution set. We thus lay the ground for a planned future paper which will investigate under which conditions the curvature polynomial (or Euler characteristic) is well approximated by its expectation.

We also remark that Theorem 1.1 can be quickly derived from the knowledge of the expected Euler characteristic of a random projective hypersurface $\mathcal{Z}(f)$ (for an invariant Gaussian f). This proof—which we found first—is again based on the kinematic formula and the generalized Gauss-Bonnet theorem. We present it in §6.

Finally, we remark that there is some connection of our work to the the study of the geometric properties of random fields. Indeed, a random polyomial $f(X_0, \ldots, X_n)$ can be seen as a "polynomial random field" defined on $\mathbb{R}^{n+1}$ or on the sphere S^n. A central topic in Adler's book [1] is the study of the Euler characteristic (and its variation called IG characteristic) of the "excursion sets" $\{x \in D \mid f(x) \geq u\}$ where f is a real valued Gaussian random field on $\mathbb{R}^{n+1}$, $u \in \mathbb{R}$, and $D \subset \mathbb{R}^{n+1}$ a compact domain with smooth boundary. Part of the interest comes from the insight that the expected Euler characteristic is a useful approximation to the distribution of the maximum of f, cf. [1]. Worsley [43] describes some applications of the statistics of the Euler characteristic of excursion sets in $\mathbb{R}^n$ to astrophysics and medicine. Adler and Taylor [41] have considerably extended and unified the previous results on the expected Euler characteristic of excursion sets to the general framework of a centered regular Gausssian field on a compact manifold. We realized that it is possible to deduce the expected Euler characteristic of a random projective hypersurface, that is, Podkorytov's result [32], from the general result in [41, Theorem 4.1], cf. Remark 6.4. We remark that our proof is methodically quite different from the ones by Podkorytov, and Adler and Taylor. Both of them rely on Morse theory, while we analyze expected tube volumes. As already mentioned before, the direct application of Morse theory does not seem feasible for the probabilistic analysis of projective varieties of higher codimension.

The structure of the paper is roughly as follows: In §2 we recall relevant facts from differential and integral geometry. Sections 3–4 prepare for the proof of Theorem 1.1, which is then given in §5. Hereby, §3 develops the necessary facts about Gaussian random vectors and symmetric matrices that are invariant under the action of the orthogonal group. In §4 we give a discussion on invariant random polynomials. In §6 we show how to quickly derive Theorem 1.1 from the knowledge of the expected Euler characteristic of a random projective hypersurface for an invariant centered Gaussian random polynomial.

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2 Background from differential and integral geometry

2.1 Weyl's tube formula and curvature polynomials in spheres

For the following material from differential geometry we refer e.g. to [37, 38]. Let M be a compact smooth m-dimensional submanifold of S^n, interpreted as a Riemannian submanifold. We denote by T_xM the tangent space of M at a point $x \in M$ and write $T_xM^\perp$ for its orthogonal complement in T_xS^n. The curvature of M at x is described by the *second fundamental form* of M at x, which is a trilinear map $\mathrm{II}_M(x)\colon T_xM \times T_xM \times T_xM^\perp \to \mathbb{R}$ that is symmetric in the first two components. In terms of local coordinates, II_M can be described as follows: let $u = (u_1, \ldots, u_m) \mapsto \varphi(u_1, \ldots, u_m) \in M \subset \mathbb{R}^{n+1}$ be a local parametrization of M. Then, for any unit normal vector $\nu \in T_xM^\perp$,

$$\mathrm{II}_M(x)\big(\partial_\alpha, \partial_\beta, \nu\big) = \big\langle \partial^2_{\alpha,\beta}\varphi, \nu\big\rangle, \tag{6}$$

using the shorthand notation $\partial_\alpha := \partial_{u_\alpha}$ and $\partial^2_{\alpha,\beta} := \partial^2_{u_\alpha,u_\beta}$. By the *Weingarten map* of M at x in direction ν we understand the self adjoint linear map $L_M(x,\nu)\colon T_xM \to T_xM$ characterized by

$$\langle L_M(x,\nu)(V), W\rangle = \mathrm{II}_M(x)(V,W,\nu) \quad \text{for } V,W \in T_xM. \tag{7}$$

In a seminal work, Weyl [42] determined the volume of the *tube*

$$T(M,\alpha) := \{y \in S^n \mid \mathrm{dist}(y,M) \le \alpha\}$$

around M for a sufficiently small radius $\alpha > 0$. He proved that

$$\mathrm{vol}(T(M,\alpha)) = \int_{x\in M}\int_{\rho=0}^{a}\int_{\nu\in S(x)} \frac{\rho^{\mu-1}\det(\mathrm{id} - \rho L_M(x,\nu))}{(1+\rho^2)^{(n+1)/2}}\, dS(x)\, d\rho\, dM, \tag{8}$$

where $a = \tan\alpha$, $S(x)$ denotes the unit sphere in $T_xM^\perp$, and $\mu := n - m$ is the codimension of M in S^n (see also [17]). Moreoever, Weyl showed that the tube volume can be written as a linear combination

$$\mathrm{vol}(T(M,\alpha)) = \sum_{0\le e\le m,\, e \text{ even}} K_{\mu+e}(M) J_{\mu+e}(\alpha)$$

of the linearly independent functions

$$J_k(\alpha) := \int_0^\alpha (\sin\rho)^{k-1}(\cos\rho)^{n-k} d\rho \text{ for } 1 \le k \le n$$

with real coefficients $K_{\mu+e}(M)$. Let $\mathcal{O}_{n-1} := 2\pi^{n/2}/\Gamma(n/2)$ denote the $(n-1)$-dimensional volume of the unit sphere S^{n-1}. The lowest order coefficient $K_\mu(M)$ equals $\mathcal{O}_{\mu-1}\mathrm{vol}(M)$ (which is intuitively plausible). Taking for M a point we conclude that the volume of a ball of radius α in S^n equals $\mathcal{O}_{n-1}J_n(\alpha)$.

We will call $K_{\mu+e}(M)$ the *curvature coefficients of the submanifold M of S^n*. In order to justify the naming of these coefficients, we remark that, after some rescaling, $K_{\mu+e}(M)$ is an isometric invariant of the Riemannian submanifold M of S^n. More precisely, for $e > 0$,

$$\frac{\mu(\mu+2)\cdots(\mu+e-2)}{\mathcal{O}_{\mu-1}} K_{\mu+e} = \int_M k_e dM$$

with some function $k_e\colon M \to \mathbb{R}$ whose value at $x \in M$ depends only on the difference of the Riemann tensor of M and the Riemann tensor of S^n restricted to M, at x (cf. [42]). (We will not need this observation in the following.) It is important to realize that these curvature coefficients are not "absolute" invariants of the Riemannian manifold M. One can show that for a subsphere $M = S^m$ of S^n we have $K_{\mu+e}(S^m) = 0$ for $e \neq 0$.

For our purposes, it will be more useful to rescale the curvature coefficients as done by Nijenhuis [31]: we define the *curvature polynomial* $\mu(M;T)$ as in the introduction (Equations (4) and (5)). It follows from the above that the constant term of the curvature polynomial describes the volume of M: we have $\mu_0(M) = \mathcal{O}_m^{-1}\operatorname{vol}(M)$. We note that $\mu(S^m;T) = 1$ for a subsphere S^m of S^n.

The curvature polynomial also encodes the Euler characteristic of M in a simple way. The following statement can be deduced from the generalized Gauss-Bonnet theorem of Allendoerfer and Weil [2] and Herglotz [18]. We provide the proof in the appendix.

Theorem 2.1 *Let M be a compact smooth submanifold of S^n of even dimension m. Then the Euler characteristic of M can be expressed as $\chi(M) = 2\,\mu(M;1)$.*

2.2 Principal kinematic formula of integral geometry for spheres

One of the main goals of integral geometry is to compute integrals of the form $\int_G I(M \cap gN)dg$, where M and N are compact smooth submanifolds of a homogenous space with respect to the action of a Lie group G, I is some integral invariant, and the integration is with respect to the invariant measure on G. Kinematic formulas provide answers to this question in the form $\int_G I(M \cap gN)dg = \sum_k c_k I_k(M)J_k(N)$ with integral invariants I_k, J_k related to I. For a comprehensive treatment of this subject we refer to Santaló's book [35]. A unified treatment of kinematic formulas in homogeneous spaces has been given by Howard [20].

Chern [12] and Federer [15] proved a general kinematic formula for submanifolds of Euclidean space with respect to the group of motions. Nijenhuis [31] pointed out a particular elegant formulation of this kinematic formula. He observed that, after some rescaling of integral invariants, $\sum_k c_k I_k(M)J_k(N)$ can be interpreted as a reduced polynomial multiplication. This leads to a great deal of simplification in our calculations, as the formulas for the coefficients c_k turn out to be quite complicated.

For submanifolds of the sphere and the orthogonal group, the kinematic formula takes exactly the same form as for Euclidean space. An indication of this at first glance astonishing fact can be found, somewhat hidden, in Santaló [35, IV.18.3. p. 320] for the special case of the intersection of domains. Howard [20] clarified this phenomenon by establishing a general transfer theorem according to which the Chern-Federer kinematic formulas hold in all simply connected spaces of constant sectional curvature and not just in Euclidean space.

The kinematic formula for spheres allows the following beautiful formulation.

Theorem 2.2 *Let M and N be compact smooth submanifolds of S^n of the dimensions m and p, respectively, such that $m+p \geq n$. Then we have*

$$\int \mu(M \cap gN; T)dg \equiv \mu(M;T)\mu(N;T) \bmod T^{m+p-n+1},$$

where the integration is with respect to the Haar measure on the orthogonal group $O(n+1)$ scaled such that the volume of $O(n+1)$ equals 1. In particular, we have

$$\int \mu(M \cap gS^p; T)dg \equiv \mu(M;T) \bmod T^{m+p-n+1}.$$

We note that Poincaré's formula

$$\int \mathcal{O}_{m+p-n}^{-1}\mathrm{vol}(M \cap gN)dg = \mathcal{O}_m^{-1}\mathrm{vol}(M)\ \mathcal{O}_p^{-1}\mathrm{vol}(N),$$

is a special case of Theorem 2.2, obtained by comparing constant coefficients.

Weyl's tube formula and the kinematic formula immediately extend from S^n to the real projective space $\mathbb{P}^n$, using the canonical isometric 2-covering map $\pi\colon S^n \to \mathbb{P}^n$. We define the curvature polynomial of a compact submanifold M of real projective space $\mathbb{P}^n$ by

$$\mu(M;T) := \mu(\pi^{-1}(M);T)). \tag{9}$$

This definition gives the appropriate scaling, as

$$\mu_0(M) = \mu_0(\pi^{-1}(M)) = \frac{1}{\mathcal{O}_m}\mathrm{vol}(\pi^{-1}(M)) = \frac{2}{\mathcal{O}_m}\mathrm{vol}(M) = \frac{1}{\mathrm{vol}(\mathbb{P}^m)}\mathrm{vol}(M).$$

Moreover, it is easy to see that the kinematic formula of Theorem 2.2 also holds for $\mathbb{P}^n$. Finally, as $\pi\colon S^n \to \mathbb{P}^n$ is a two sheeted covering map, Theorem 2.1 implies that $\chi(M) = \frac{1}{2}\chi(\pi^{-1}(M)) = \mu(\pi^{-1}(M);1) = \mu(M;1)$.

Remark 2.3 It is possible to derive Theorem 2.2 from the kinematic formulas given in Santaló [35, (15.72), p. 269 and IV.18.3. p. 320] for the intersection of domains in Euclidean space and spheres, respectively. Santaló [35, p. 222, p. 302] assigns to a compact domain Q in $\mathbb{R}^n$ or S^n, bounded by a smooth hypersurface ∂Q, the following integral of mean curvature

$$M_i^{San}(\partial Q) := \binom{n-1}{i}^{-1} \int_{\partial Q} \sigma_i(\kappa_1, \ldots, \kappa_{n-1})\, d(\partial Q).$$

Hereby, $\sigma_i(\kappa_1, \ldots, \kappa_{n-1})$ stands for the ith elementary symmetric function in the principal curvatures κ_j of the hypersurface ∂Q. The curvature coefficients $K_{i+1}(M)$ of a a compact submanifold M of S^n can be related to Santaló's integral of mean curvature of the tubes around M as follows (define $K_j(M) = 0$ for $j < \mathrm{codim}_{S^n} M$)

$$K_{i+1}(M) = \binom{n-1}{i} \lim_{\alpha \to 0} M_i^{San}(\partial T(M, \alpha)). \tag{10}$$

(This follows as in [40, V§4], where a similar result is shown for Euclidean space.)

3 Invariant Gaussian vectors and matrices

We develop here facts about invariant random matrices that will be used in §5 for the proof of the main result.

A random vector is called centered iff its expectation is zero. We call two random vectors *equivalent* if they have the same distribution. In particular, equivalent random vectors have the same expectation and covariance matrix. We shall write $X \sim N(0, \sigma^2)$ to indicate that X is a real valued Gaussian variable with mean zero and variance σ^2.

3.1 Invariant random vectors

A random vector $V \in \mathbb{R}^n$ is called $O(n)$-*invariant* iff gV is equivalent to V for all $g \in O(n)$. For simplicity, we assume that V has a density.

Lemma 3.1 *Let $V \in \mathbb{R}^n$ be an $O(n)$-invariant random vector. Then $V/\|V\|$ is uniformly distributed in S^{n-1} and independent of $\|V\|$.*

Proof. By invariance, the density of V depends only on $\|V\|$. □

Invariant Gaussian random vectors are easy to characterize.

Lemma 3.2 *A Gaussian random vector $V = (V_1, \ldots, V_n)$ is $O(n)$-invariant iff it is centered and $V_1, \ldots, V_n$ are independent and have the same variance.*

Proof. Suppose V is $O(n)$-invariant. Then $-V$ is equivalent to V (take $g = -I_n$), hence $\mathbb{E}V = 0$. By assumption, gVV^Tg^T is equivalent to VV^T, hence the covariance matrix $A := \mathbb{E}(VV^T)$ satisfies $gAg^T = A$ for all $g \in O(n)$. A straightforward calculation shows that A must be a multiple of the unit matrix I_n (take for g

rotations in two dimensional coordinate subspaces). The converse follows from the fact that Gaussian random variables are characterized by their expectation and covariance matrix. $\square$

3.2 Invariant symmetric matrices

Let Σ_n denote the space of real symmetric n by n matrices. The Frobenius norm of $W \in \Sigma_n$ is defined as $\|W\|_F := (\sum_{i,j} W_{ij}^2)^{1/2}$. We will assume $n > 1$.

Definition 3.3 The *parameter* $\delta(W)$ of a random matrix $W \in \Sigma_n$ is defined as

$$\delta(W) := \frac{1}{n(n-1)}\big(\mathbb{E}\,(\mathrm{tr}W)^2 - \mathbb{E}\,\|W\|_F^2\big).$$

A random matrix $W \in \Sigma_n$ is called *$O(n)$-invariant* iff gWg^T is equivalent to W for all $g \in O(n)$.

The following proposition classifies all invariant Gaussian $W \in \Sigma_n$.

Proposition 3.4 *Suppose $T \in \Sigma_n$ with independent T_{ij} and $T_{ij} \sim N(0,1)$ for $i \neq j$ and $T_{ii} \sim N(0,2)$, $n > 1$. Moreover, let $r, s \in \mathbb{R}$ and $Z \sim N(0,1)$ be independent of T. Then $W = rZI_n + sT$ is $O(n)$-invariant and has the parameter $\delta(W) = r^2 - s^2$. Moreover, any $O(n)$-invariant Gaussian $W \in \Sigma_n$ is equivalent to one of this form, in particular $\mathbb{E}\,W = 0$.*

Proof. In order to see that T is $O(n)$-invariant it is sufficient to check that $\widetilde{T} = gTg^T$ is equivalent to T for a set of generators g. Since $\widetilde{T}$ and T are both centered Gaussian it is sufficient to check that they have the same covariance matrix. The group $O(n)$ is generated by $\mathrm{diag}(-1, 1, \ldots, 1)$ and the rotations in two dimensional coordinate spaces. Invariance under the latter is verified by a straightforward calculation of covariances. Moreover, a calculation shows that

$$\mathbb{E}(\mathrm{tr}W)^2 = n^2r^2 + 2ns^2, \quad \mathbb{E}\,\|W\|_F^2 = nr^2 + n(n+1)s^2.$$

This implies $\delta(W) = r^2 - s^2$.

Suppose now that W is $O(n)$-invariant. Let g be the product of the permutation matrix $P_{\pi^{-1}}$ and $\mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$ where $\varepsilon_i = \pm 1$. Then $gWg^T = (\varepsilon_i\varepsilon_j W_{\pi(i),\pi(j)})$ is equivalent to W. Taking $\pi = \mathrm{id}, \varepsilon_1 = -1, \varepsilon_2 = \ldots \varepsilon_n = 1$ we conclude that (W_{11}, W_{12}) is equivalent to $(W_{11}, -W_{12})$, hence $\mathbb{E}\,W_{12} = 0$ and $\mathbb{E}\,(W_{11}W_{12}) = 0$. By choosing appropriate π and ε_i one can similarly show that $\mathbb{E}\,(W_{ij}W_{k\ell}) = 0$ except in the cases where $i = j$ and $k = \ell$ (details are left to the reader). Similarly, one shows that W is centered. Moreover, by conjugating with permutation matrices, we see that

$$\mathbb{E}\,W_{ii}^2 = \mathbb{E}\,W_{11}^2,\ \mathbb{E}\,W_{ij}^2 = \mathbb{E}\,W_{12}^2 =: s^2,\ \mathbb{E}\,(W_{ii}W_{jj}) = \mathbb{E}\,(W_{11}W_{22}) =: r^2 \quad (i \neq j).$$

If we can prove that

$$\mathbb{E}W_{11}^2 = \mathbb{E}\,(W_{11}W_{22}) + 2\,\mathbb{E}W_{12}^2, \tag{11}$$

then W has the same covariance matrix as $rZI_n + sT$ and we are done.

For showing (11) suppose without loss of generality $n = 2$. Put $\widetilde{W} = gWg^T$ where $g = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$. Then $\widetilde{W}_{11} = W_{11}\cos^2\varphi + W_{22}\sin^2\varphi - 2W_{12}\cos\varphi\sin\varphi$. Hence

$$\mathbb{E}\widetilde{W}_{11}^2 = \mathbb{E}\,W_{11}^2\,(\cos^4\varphi + \sin^4\varphi) + 2\left(2\,\mathbb{E}\,W_{12}^2 + \mathbb{E}\,(W_{11}W_{22})\right)\cos^2\varphi\sin^2\varphi,$$

Using that $\cos^4\varphi + \sin^4\varphi = 1 - 2\,\cos^2\varphi\sin^2\varphi$, the assertion (11) follows. □

Lemma 3.5 *Suppose $W, W_1, \ldots, W_s \in \Sigma_n$ are random symmetric matrices and $\lambda_1, \ldots, \lambda_s \in \mathbb{R}$. Further, let $u \in \mathbb{R}$ be a random variable. Then:*

(i) $\delta(W + uI_n) = \delta(W) + \mathbb{E}\,u^2 + \frac{2}{n}\mathbb{E}\,(u\,\mathrm{tr}W)$.

(ii) $\delta(\lambda_1 W_1 + \cdots + \lambda_s W_s) = \lambda_1^2\delta(W_1) + \cdots + \lambda_s^2\delta(W_s)$ *if $W_1, \ldots, W_s$ are independent.*

Proof. (i) Put $\widetilde{W} = W + uI_n$. Then $(\mathrm{tr}\widetilde{W})^2 = (\mathrm{tr}\,W)^2 + 2nu\,\mathrm{tr}\,W + n^2u^2$. Moreover, $\|\widetilde{W}\|_F^2 = \|W\|_F^2 + 2u\,\mathrm{tr}\,W + nu^2$. The first assertion follows.

(ii) Put $W := \sum_\sigma \lambda_\sigma W_\sigma$. We have $(\mathrm{tr}W)^2 = \sum_{\sigma,\tau}\lambda_\sigma\lambda_\tau\ (\mathrm{tr}W_\sigma)(\mathrm{tr}W_\tau)$. By indepence and since W_σ are centered we get $\mathbb{E}\,(\mathrm{tr}W)^2 = \sum_\sigma \lambda_\sigma^2\ \mathbb{E}\,(\mathrm{tr}W_\sigma)^2$. Similarly, we obtain $\mathbb{E}W_{k\ell}^2 = \sum_\sigma \lambda_\sigma^2\ \mathbb{E}\,((W_\sigma)_{k\ell})^2$. Hence $\mathbb{E}\,\|W\|_F^2 = \sum_\sigma \lambda_\sigma^2\ \|W_\sigma\|_F^2$. The second assertion follows. □

Here is a further result stating that stochastic independence follows from invariance.

Lemma 3.6 *Consider a random $(u, V, W) \in \mathbb{R} \times \mathbb{R}^n \times \Sigma_n$ with joint Gaussian distribution such that (u, gV, gWg^T) is equivalent to (u, V, W), for all $g \in O(n)$. (We will call such (u, V, W) $O(n)$-invariant.) Then:*

(i) *V is independent of u and W.*

(ii) *If $\mathbb{E}\,(u\,\mathrm{tr}\,W) = 0$, then u and W are independent.*

Proof. V and W are centered by invariance and we may assume w.l.o.g. that u is centered. (i) Taking $g = -I_n$ we see that $(u, -V, W)$ is equivalent to (u, V, W). Hence $\mathbb{E}\,(uV_i) = 0$ and $\mathbb{E}\,(V_iW_{jk}) = 0$.

(ii) We argue similarly as in the proof of Proposition 3.4. Using that (u, gWg^T) is equivalent to (u, W) for $g = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_i = \pm 1$ we get $\mathbb{E}\,(uW_{ij}) = 0$ for $i \neq j$. By taking for g a permutation matrix we conclude $\mathbb{E}\,(uW_{ii}) = \mathbb{E}\,(uW_{11})$. By assumption, $\mathbb{E}\,(uW_{11}) = n^{-1}\mathbb{E}\,(u\,\mathrm{tr}\,W) = 0$. Hence u is uncorrelated with all W_{ij}. □

Corollary 3.7 *Consider a random* $(u, W) \in \mathbb{R} \times \Sigma_n$ *with joint Gaussian distribution such that* (u, gWg^T) *is equivalent to* (u, W)*, for all* $g \in O(n)$*. Let* W_c *denote the random matrix* W *conditioned on* $u = 0$*. Then* W_c *is* $O(n)$*-invariant Gaussian and has the following parameter*

$$\delta(W_c) = \delta(W) - \frac{(\mathbb{E}\,(u\,\mathrm{tr}\,W))^2}{n^2\,\mathbb{E}\,u^2}.$$

Proof. Consider the random matrix $\widetilde{W} := W - \lambda u I_n$, where $\lambda := \frac{\mathbb{E}\,(u\,\mathrm{tr}\,W)}{n\,\mathbb{E}\,u^2}$. Then $\mathbb{E}\,(u\,\mathrm{tr}\widetilde{W}) = 0$. Moreover, $(u, \widetilde{W})$ is $O(n)$-invariant Gaussian. According to Lemma 3.6, u and $\widetilde{W}$ are independent. Hence W_c has the same distribution as $\widetilde{W}$. Moreover, by Lemma 3.5,

$$\delta(\widetilde{W}) = \delta(W) + \lambda^2 \mathbb{E}\,u^2 - \frac{2\lambda}{n}\,\mathbb{E}\,(u\,\mathrm{tr}\,W) = \delta(W) - \lambda^2\,\mathbb{E}\,u^2 = \delta(W) - \frac{(\mathbb{E}\,(u\,\mathrm{tr}\,W))^2}{n^2\,\mathbb{E}\,u^2}$$

as claimed. □

3.3 Expected determinant of random symmetric matrices

This section is devoted to the proof of the following result. It is stated in [32], but the proof is only vaguely sketched there.

Proposition 3.8 *Suppose* $W \in \Sigma_n$ *is* $O(n)$*-invariant and Gaussian,* $n > 1$*. Then we have*

$$\mathbb{E}\,\det(I_n + W) = \mathbb{E}\big(1 + \sqrt{\delta(W)}\,X\big)^n,$$

where $X \sim N(0,1)$ *is a standard normal random variable. (Note that* $\delta(W)$ *may be negative.)*

To prepare for the proof we denote by ($k \in \mathbb{N}$)

$$\gamma_k = \mathbb{E}\,|X|^k = \frac{2}{\sqrt{2\pi}} \int_0^\infty y^k e^{-\frac{y^2}{2}}\,dy = \frac{1}{\sqrt{\pi}}\,2^{k/2}\,\Gamma(\frac{k+1}{2}) \tag{12}$$

the k-th absolute moment of a standard normal random variable $X \sim N(0,1)$. In particular,

$$\gamma_{2m} = 1 \cdot 3 \cdot 5 \cdots (2m-1) = (2m)!/(2^m m!), \quad \gamma_k \mathcal{O}_k = 2(2\pi)^{k/2}. \tag{13}$$

The following lemma about the higher moments of Gaussian vectors is well known, cf. [1, p. 108].

Lemma 3.9 *Let* $(Y_1, \ldots, Y_n)$ *be a centered Gaussian vector. Then* $\mathbb{E}(Y_1 \cdots Y_n) = 0$ *if* n *is odd. Otherwise, if* $n = 2m$*,*

$$\mathbb{E}(Y_1 \cdots Y_{2m}) = \sum \mathbb{E}(Y_{i_1} Y_{i_2}) \cdots \mathbb{E}(Y_{i_{2m-1}} Y_{i_{2m}}),$$

where the sum is over all $\gamma_{2m} = (2m)!/(2^m m!)$ *different ways of grouping* $Y_1, \ldots, Y_{2m}$ *into* m *pairs.*

Clearly, the lemma implies that $\mathbb{E}\det W = 0$ for any centered Gaussian matrix $W \in \mathbb{R}^{n\times n}$ if n is odd.

Corollary 3.10 *The random matrix $T \in \Sigma_n$ of Proposition 3.4 satisfies $\mathbb{E}\det T = (-1)^m \gamma_{2m}$ if $n = 2m$ and $\mathbb{E}\det T = 0$ otherwise.*

PROOF. We have $\mathbb{E}\det T = \sum_\pi \operatorname{sgn}(\pi)\,\mathbb{E}(T_{1,\pi(1)}\cdots T_{2m,\pi(2m)})$. Lemma 3.9 and the fact that the entries of T are independent imply that only the products of transpositions of the form $\pi = (i_1 j_1)\cdots(i_m j_m)$ with $\{i_1, j_1, \ldots, i_m, j_m\} = \{1, 2, \ldots, 2m\}$ contribute to the sum. The contribution of each such π is $(-1)^m$ and there are γ_{2m} such π. $\square$

PROOF OF PROPOSITION 3.8. According to Proposition 3.4 we may assume $W = rZI_n + sT$. Then $\delta(W) = r^2 - s^2$. By expanding the determinant it is easy to see that

$$\det(I_n + W) = \det((1 + rZ)I_n + sT) = \sum_K (1 + rZ)^{n-|K|}\, s^{|K|}\, \det T_{K,K},$$

where the sum is over all subsets K of $\{1, 2, \ldots, n\}$ and $T_{K,K}$ denotes the principal submatrix obtained by selecting the rows and columns in K. Using Corollary 3.10 and taking into account the independence of Z and T we obtain

$$\mathbb{E}\det(I_n + W) = \sum_{j=0}^{\lfloor n/2\rfloor} \binom{n}{2j} \mathbb{E}\,(1 + rZ)^{n-2j}\; s^{2j}(-1)^j\gamma_{2j}.$$

Suppose $Y \sim N(0,1)$ is independent of Z. Using $i = \sqrt{-1}$ we have $(-1)^j\gamma_{2j} = \mathbb{E}(iY)^{2j}$ and hence

$$\begin{aligned} \det(I_n + W) &= \sum_{j=0}^{\lfloor n/2\rfloor} \binom{n}{2j} \mathbb{E}\,(1 + rZ)^{n-2j}\, \mathbb{E}(isY)^{2j} \\ &= \mathbb{E}(1 + rZ + isY)^n \\ &= \sum_{j=0}^{\lfloor n/2\rfloor} \binom{n}{2j} \mathbb{E}\,(rZ + isY)^{2j}. \end{aligned}$$

On the other hand, using that $\binom{2j}{2k}\gamma_{2k}\gamma_{2j-2k} = \binom{j}{k}\gamma_{2j}$ for $0 \le k \le j$, we get

$$\begin{aligned} \mathbb{E}\,(rZ + isY)^{2j} &= \sum_{k=0}^{j} \binom{2j}{2k} r^{2k}\gamma_{2k}\,(-1)^{j-k}\, s^{2j-2k}\,\gamma_{2j-2k} \\ &= \gamma_{2j}\sum_{k=0}^{j}\binom{j}{k} r^{2k}\,(-1)^{j-k}\, s^{2j-2k} = \gamma_{2j}(r^2 - s^2)^j. \end{aligned}$$

We conclude that

$$\det(I_n + W) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \mathbb{E}\,(\sqrt{r^2 - s^2}Z)^{2j} = \mathbb{E}\,(1 + \sqrt{r^2 - s^2}Z)^n,$$

which finishes the proof. □

3.4 Two auxiliary results on random matrices

Let $A \in \mathbb{R}^{s \times n}$ be a matrix of rank $s \leq n$. We denote by $A_m \in \mathbb{R}^n$ the m-th row of A and by $A_m^\perp$ the orthogonal projection of A_m onto the space spanned by $A_1, \ldots, A_{m-1}$. Let $\mathrm{vol}(A_1, \ldots, A_m)$ denote the volume of the parallelepiped in $\mathbb{R}^n$ spanned by $A_1, \ldots, A_m$. Clearly, $\mathrm{vol}(A_1, \ldots, A_m) = \mathrm{vol}(A_1, \ldots, A_{m-1})\, \|A_m^\perp\|$.

Lemma 3.11 *Suppose that $A \in \mathbb{R}^{s \times n}$ is a random matrix with independent standard Gaussian entries. Then we have for $1 \leq m \leq n$ and $0 \leq j \leq \lfloor \frac{n-m}{2} \rfloor$*

$$\mathbb{E}_A \left(\mathrm{vol}(A_1, \ldots, A_{m-1}) \cdot \frac{1}{\|A_m^\perp\|^{2j-1}} \right) = \frac{\mathcal{O}_{n-m}}{2(2\pi)^{\frac{n-m}{2}}} \, \frac{\gamma_n \gamma_{n-m+1-2j}}{\gamma_{n-m+1}}.$$

Proof. The invariance under rotations of the standard normal distribution shows that, conditioning on $A_1, \ldots, A_{m-1}$, the random variable $A_m^\perp$ has a standard normal distribution in the orthogonal complement of $A_1, \ldots, A_{m-1}$ in $\mathbb{R}^n$, which is dimension $n - m + 1$ with probability one. Hence $\mathbb{E}\left[\|A_m^\perp\| \mid A_1, \ldots, A_{m-1}\right] = K_{n-m+1}$, where we have written $K_d := \mathbb{E}\,\|X\|$ for a standard normal X in $\mathbb{R}^d$. An elementary computation gives (cf. [3])

$$K_d = \sqrt{2}\,\frac{\Gamma((d+1)/2)}{\Gamma(d/2)}, \quad K_1 K_2 \cdots K_m = \frac{1}{\sqrt{\pi}}\, 2^{\frac{m}{2}}\; \Gamma(\frac{m+1}{2}) = \gamma_m.$$

Put $\mathrm{vol}_m(A) := \mathrm{vol}(A_1, \ldots, A_m)$. Then $\mathrm{vol}_m(A) = \mathrm{vol}_{m-1}(A)\, \|A_m^\perp\|$. Hence

$$\mathbb{E}[\mathrm{vol}_m(A)] = \mathbb{E}\Big[\mathrm{vol}_{m-1}(A)\; \mathbb{E}\left[\|A_m^\perp\| \mid A_1, \ldots, A_{m-1}\right]\Big] = \mathbb{E}[\mathrm{vol}_{m-1}(A)]\; K_{n-m+1},$$

which implies

$$\mathbb{E}[\mathrm{vol}_m(A)] = K_{n-m+1} K_{n-m+2} \cdots K_n = \frac{\gamma_n}{\gamma_{n-m}}. \tag{14}$$

Using polar coordinates, we get for almost all values of $A_1, \ldots, A_{m-1}$

$$\begin{aligned} \mathbb{E}_{A_m}\left[\|A_m^\perp\|^{1-2j} \mid A_1, \ldots, A_{m-1}\right] &= \frac{\mathcal{O}_{n-m}}{(2\pi)^{\frac{n-m+1}{2}}} \int_0^\infty r^{n-m-2j+1} e^{-\frac{r^2}{2}}\, dr \\ &= \frac{1}{2}\, \frac{\mathcal{O}_{n-m}}{(2\pi)^{\frac{n-m}{2}}}\, \gamma_{n-m-2j+1}. \end{aligned}$$

The claim follows by combining this with (14). □

The Borel-Moore inverse of a matrix $A \in \mathbb{R}^{s\times n}$ of rank s is defined by $A^\dagger := A^T(AA^T)^{-1} \in \mathbb{R}^{n\times s}$, cf. [4]. It is characterized by the following properties: $AA^\dagger = I_s$ and $A^\dagger A$ is the orthogonal projection onto the orthogonal complement $(\ker A)^\perp$ of the kernel of A. Note that $(\ker A)^\perp$ is generated by the rows of A. Let us denote by $\widetilde{A}$ the restriction of the linear map $A\colon \mathbb{R}^n \to \mathbb{R}^s$ to $(\ker A)^\perp$. Then the linear map $A^\dagger\colon \mathbb{R}^s \to \mathbb{R}^n$ is the inverse of $\widetilde{A}$. We note that

$$|\det \widetilde{A}| = \mathrm{vol}(A_1,\ldots,A_s) = \sqrt{\det(AA^T)}. \tag{15}$$

(Proof: $\widetilde{A}^T e_i = A_i$, hence $|\det \widetilde{A}^T| = \mathrm{vol}(A_1,\ldots,A_s)$. It is well known that the latter equals $\sqrt{\det AA^T}$.)

We denote by $(N_1,\ldots,N_s)$ the orthonormal basis of $(\ker A)^\perp$ obtained from the rows of A by Gram-Schmidt orthogonalization. This defines the orthogonal map

$$Q_A\colon \mathbb{R}^s \to (\ker A)^\perp,\ u \mapsto \sum_{\sigma=1}^{s} u_\sigma N_\sigma. \tag{16}$$

By composing Q_A with the adjoint $(A^\dagger)^T$ of the Borel-Moore inverse $A^\dagger$ we obtain the linear endomorphism $(A^\dagger)^T Q_A\colon \mathbb{R}^s \to \mathbb{R}^s$.

The following is the main result of this subsection. It will play crucial role in the proof of Theorem 1.1.

Proposition 3.12 *Suppose that $A \in \mathbb{R}^{s\times n}$ is a random matrix with independent standard Gaussian entries. Let $u \in S^{s-1}$ be a uniformly distributed random unit vector which is independent of A. Then the random variable*

$$\left(\sqrt{\det AA^T},\, (A^\dagger)^T Q_A(u)\right)$$

with values in $\mathbb{R}\times\mathbb{R}^s$ has the same distribution as

$$\left(\mathrm{vol}(A_1,\ldots,A_{s-1})\cdot \|A_s^\perp\|,\, \frac{1}{\|A_s^\perp\|}\cdot w\right)$$

where w is uniformly distributed in the sphere S^{s-1} and independent of A.

We give some preparations for the proof. Let $A \in \mathbb{R}^{s\times n}$ be of rank s. Pick $y \in (\ker A)^\perp$ and define $A^{(y)}$ as the restriction of A to the orthogonal complement of $\mathbb{R}y \oplus \ker A$. This can be described by the following commutative diagram (where upgoing arrows denote inclusions):

$$\begin{array}{ccc}
(\ker A)^\perp & \xrightarrow{\widetilde{A}} & \mathbb{R}^s \\
\uparrow & & \uparrow \\
(\mathbb{R}y \oplus \ker A)^\perp & \xrightarrow{A^{(y)}} & \mathrm{im}\, A^{(y)}
\end{array}$$

Suppose now $\|y\| = 1$ and let $(Ay)^\perp$ denote the orthogonal projection of Ay onto the image of $A^{(y)}$. Then it is easy to see that

$$|\det \widetilde{A}| = |\det A^{(y)}| \cdot \|(Ay)^\perp\|. \tag{17}$$

Lemma 3.13 *Let $A \in \mathbb{R}^{s\times n}$ be of rank $s \leq n$ and $y \in (\ker A)^\perp$ be a unit vector. Then the length of the vector $v = (A^\dagger)^T y \in \mathbb{R}^s$ can be described as*

$$\|v\| = \left|\frac{\det A^{(y)}}{\det \widetilde{A}}\right| = \frac{1}{\|(Ay)^\perp\|}.$$

Proof. We have $v^T A = y^T A^\dagger A = y^T$, hence $A^T v = y$. Thus v can be interpreted as the unique solution of the system of linear equations

$$v_1 A_1 + \cdots v_s A_s = y.$$

Cramer's rule implies that

$$|v_i| = \frac{\mathrm{vol}(A_1, \ldots, A_{i-1}, y, A_{i+1}, \ldots, A_s)}{\mathrm{vol}(A_1, \ldots, A_s)}.$$

Let $\overline{A}_i$ denote the projection of A_i onto the orthogonal complement of $\mathbb{R}y$. Then

$$\mathrm{vol}(A_1, \ldots, A_{i-1}, y, A_{i+1}, \ldots, A_s) = \mathrm{vol}(\overline{A}_1, \ldots, \overline{A}_{i-1}, \overline{A}_{i+1}, \ldots, \overline{A}_s).$$

It is now sufficient to prove that (cf. (15))

$$\sum_{i=1}^{s} \mathrm{vol}(\overline{A}_1, \ldots, \overline{A}_{i-1}, \overline{A}_{i+1}, \ldots, \overline{A}_s)^2 = (\det A^{(y)})^2. \tag{18}$$

By an orthogonal transformation, we may assume without loss of generality that $\ker A = 0 \times \mathbb{R}^{n-s}$ and $y = (0, \ldots, 0, 1, 0, \ldots, 0)$ (with 1 at position s). Write $A = (a_{ij})$. Then $a_{ij} = 0$ for $j > s$ and $\widetilde{A}$ is given by the matrix $(a_{ij})_{i,j\leq s}$ with respect to the canonical bases. Moreover, $A^{(y)}$ is given by the matrix $M := (a_{ij})_{i\leq s, j<s}$ in $\mathbb{R}^{s\times(s-1)}$. The relation of Binet-Cauchy [4] states that

$$\sum_{i=1}^{s} (\det M_i)^2 = \det(M^T M),$$

where M_i stands for the square matrix obtained from M by deleting the i-th row. This is exactly the asserted Equation (18). $\square$

Proof of Proposition 3.12. Consider the following random process. First choose $A \in \mathbb{R}^{s\times n}$ at random with independent standard Gaussian entries. Then pick a unit vector $y \in (\ker A)^\perp$ uniformly at random. The determinant of the Jacobian of the transformation $(A,u) \mapsto (Q_A(u), A)$ of $\mathbb{R}^{s\times n} \times S^{s-1}$ has absolute value 1, since $|\det Q_A| = 1$. Hence this transformation preserves the product of the Gaussian measure with the uniform measure. We conclude that the random variable $\big(\sqrt{\det AA^T}, (A^\dagger)^T Q_A(u)\big)$ is equivalent to $\big(\sqrt{\det AA^T}, (A^\dagger)^T(y)\big)$.

Equations (15) and (17) give that

$$\sqrt{\det(AA^T)} = |\det \widetilde{A}| = |\det A^{(y)}| \cdot \|(Ay)^\perp\|.$$

Lemma 3.13 implies that

$$(A^\dagger)^T(y) = \frac{1}{\|(Ay)^\perp\|} \cdot w$$

with a random unit vector $w \in S^{s-1}$. It is clear that $(A^\dagger)^T(y)$ is $O(s)$-invariant. According to Lemma 3.1, this implies that w is uniformly distributed and independent of $\|(Ay)^\perp\|$.

As in the proof of Lemma 3.11, we see that $\|A_s^\perp\|$ conditioned on $A_1, \dots, A_{s-1}$ is equivalent to $\|X\|$, where X is standard normal in $\mathbb{R}^{n-s+1}$. It is not hard to see that $\|(Ay)^\perp\|$ has the same distribution as $\|X\|$. Similarly, one shows that $|\det A^{(y)}|$ is equivalent to $\mathrm{vol}(A_1, \dots, A_{s-1})$. (Note that if $g_1, \dots, g_{s-1}, y$ is an orthonormal basis of $(\ker A)^\perp$, then $|\det A^{(y)}| = \mathrm{vol}(Ag_1, \dots, Ag_{s-1})$.) Finally, $\|(Ay)^\perp\|$ is independent of $|\det A^{(y)}|$. $\square$

4 Invariant random polynomials

4.1 Classification

We briefly describe the classification of invariant Gaussian polynomials and refer to Kostlan [23, 24] for more details. This classification is just for illustration and will not be needed for the proof of Theorem 1.1.

Recall that $H_{d,n}$ denotes the vector space of homogeneous real polynomials of degree d in the variables $X_0, \dots, X_n$. The orthogonal group $O(n+1)$ operates on $H_{d,n}$ (from the left) in the natural way: for $f \in H_{d,n}$ and $g \in O(n+1)$ we set $(gf)(X) := f(g^{-1}X)$, where $X = (X_0, \dots, X_n)^\top$.

Definition 4.1 A random polynomial $f \in H_{d,n}$ is called $O(n+1)$-*invariant* (invariant for short) iff gf is equivalent to f for all all $g \in O(n+1)$.

Remark 4.2 Any invariant random polynomial $f \in H_{d,n}$ is centered if d is odd. Moreover, if d is even, it is easy to reduce to the case where f is centered, cf. [24, §5.3]. For convenience we will therefore additionally require that f is centered.

Example 4.3 The most natural example of an invariant random polynomial in $H_{d,n}$ is obtained as follows. Write $f \in H_{d,n}$ in the form $f = \sum_\alpha f_\alpha X^\alpha$, where the sum is over all multiindices $\alpha \in \mathbb{N}^{n+1}$ such that $|\alpha| := \sum_i \alpha_i = d$. Assume that the coefficients f_α are independent with centered Gaussian distribution and variance $\mathrm{Var}(f_\alpha) = \binom{d}{\alpha} := \frac{d!}{\alpha_0! \cdots \alpha_n!}$. The covariance function $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}, (x,y) \mapsto \mathbb{E}\,(f(x)f(y))$ satisfies

$$\mathbb{E}\,(f(x)f(y)) = \sum_{\alpha,\beta} \mathbb{E}(f_\alpha f_\beta) x^\alpha y^\beta = \sum_\alpha \binom{d}{\alpha} x^\alpha y^\alpha = (\sum_{i=0}^n x_i y_i)^d = \langle x, y\rangle^d.$$

and it is thus invariant under the action of $O(n+1)$. It follows that f is an invariant Gaussian random polynomial. We will say that f is *Kostlan distributed* [23]. Up to a scalar, this random polynomial can be characterized by requiring that $f \in H_{d,n}$ is invariant and centered and has stochastically independent coefficients f_α (cf. [23, Theorem 4.5]). It is interesting that the normal distribution is enforced by the above requirements only.

It is possible to characterize all invariant centered Gaussian random polynomials of $H_{d,n}$. For doing so, it is helpful to start with some general observations.

A centered Gaussian distribution on $V = \mathbb{R}^n$ is characterized by its covariance matrix. In coordinate free language, this corresponds to the choice of an inner product on the dual space V^* defined by

$$V^* \times V^* \to \mathbb{R},\ (\lambda, \mu) \mapsto \mathbb{E}_v\,(\lambda(v)\mu(v))$$

Consider now the general situation of a compact Lie group G operating on a real vector space V (in our situation $G = O(n+1)$). It is easy to see that a centered Gaussian distribution on V is G-invariant iff the corresponding inner product is G-invariant, that is, $\langle gu, gv\rangle = \langle u, v\rangle$ for all $g \in G$ and all $u, v \in V$.

From the representation theory of compact groups it is known that

1. On an irreducible G-module W there is a (up to a positive scalar) unique G-invariant inner product.
2. Any two nonisomorphic submodules of V are orthogonal with respect to a G-invariant inner product of V.

(The first statement follows from Schur's lemma; for the second see [19, §27, p. 29].)

Suppose that V splits into a direct sum $V = \oplus_{i=1}^m W_i$ of pairwise nonisomorphic irreducible submodules W_i. Choose an invariant inner product $\langle\ ,\ \rangle_i$ on each W_i. Then, according to the above facts, the invariant inner products on V are of the form

$$V \times V \to \mathbb{R}, (\oplus u_i, \oplus_i v_i) \mapsto \textstyle\sum_{i=1}^m c_i \langle u_i, v_i\rangle_i,$$

parameterized by $c_1, \ldots, c_m > 0$.

We briefly describe the decomposition of the $O(n+1)$-module $H_{d,n}$ into irreducible submodules, cf. [16, §5.2.3]. Write $r^2 := \sum_{i=0}^{n} X_i^2$ and consider the equivariant linear map $\varphi\colon H_{d,n} \to H_{d,n}$, $f \mapsto r^2 \Delta f$ arising from the Laplace operator Δ on $\mathbb{R}^{n+1}$. It is known that the kernel $\mathcal{H}_{d,n}$ is irreducible. The elements of $\mathcal{H}_{d,n}$ are called *harmonic polynomials* of degree d. The map φ has the following eigenspace decomposition

$$H_{d,n} = \bigoplus_{i=0}^{\lfloor d/2 \rfloor} r^2 \mathcal{H}_{d-2i,n}, \tag{19}$$

which is thus a decomposition of $H_{d,n}$ into nonisomorphic irreducible submodules ($\mathcal{H}_{d-2i,n}$ corresponds to the eigenvalue $2i(n+2d-2i-1)$). We conclude that the invariant inner products on $H_{d,n}$ can be parameterized by $\lfloor d/2 \rfloor$ positive numbers. We refrain from explicitly describing these inner products and refer instead to [24] for details. To summarize, we see that the invariant centered Gaussian random polynomials of $H_{d,n}$ can be parameterized by $\lfloor d/2 \rfloor$ positive numbers.

4.2 Parameter of invariant random polynomials

A polynomial $f \in H_{d,n}$ defines a differentiable map $f\colon S^n \to \mathbb{R}$. We denote by $Df(x)\colon T_x S^n \to \mathbb{R}$ and $D^2 f(x)\colon T_x S^n \times T_x S^n \to \mathbb{R}$ the first and second order derivative of f at $x \in S^n$. They are characterized by

$$f\left(\frac{x+\lambda V}{\|x+\lambda V\|}\right) = f(x) + \lambda Df(x)(V) + \frac{\lambda^2}{2} D^2 f(x)(V,V) + o(\lambda^2) \tag{20}$$

for $V \in T_x S^n$ and $\lambda \in \mathbb{R}$ going to zero. At the point $q = (1, 0 \ldots, 0)$ we can identify the tangent space $T_q S^n = 0 \times \mathbb{R}^n$ with $\mathbb{R}^n$. Clearly, $Df(q)$ is determined by the gradient vector $(\partial_1 f(q), \ldots, \partial_n f(q))$.

Definition 4.4 The *parameter* $\delta(f)$ of an invariant random polynomial $f \in H_{d,n}$ is defined as

$$\delta(f) := \frac{\|x\|^2\, \mathbb{E}\, \|Df(x)\|^2}{n \mathbb{E}\, f(x)^2}$$

where $x \in \mathbb{R}^{n+1} \setminus \{0\}$ (this is independent of x by invariance and homogenity).

Remark 4.5

1. If $f \in H_{d,n}$ is invariant then $Df(q)$ is Gaussian with covariance matrix $\delta(f)\, \mathbb{E}\, f(q)^2\, I_n$. Thus $\delta(f) = \frac{\mathbb{E}\, (\partial_k f(q))^2}{\mathbb{E}\, f(q)^2}$ for any $1 \le k \le n$.
2. The parameter of the Kostlan distribution equals the degree d.

Proof. The first statement follows from Lemma 3.2. For the second just note that $f(q) = f_{(d,0,\ldots,0)}$ and $\partial_{X_1} f(q) = f_{(d-1,1,0,\ldots,0)}$. □

Lemma 4.6 *Suppose $f \in H_{d,n}$ is $O(n+1)$-invariant and $n' \leq n$. Then the restriction $f' \in H_{d,n'}$ of f to $\mathbb{R}^{n'+1}$ is $O(n'+1)$-invariant and has the same parameter, i.e., $\delta(f') = \delta(f)$. Moreover, if f is Kostlan distributed, then so is f'.*

PROOF. It is clear that the distribution of f' is invariant with respect to the action of $O(n'+1)$. In order to see that the parameter remains the same just note that restricting f to $\mathbb{R}^{n'+1}$ means substituting the variables $X_{n'+1}, \ldots, X_n$ by 0. The assertion about the Kostlan distribution is obvious. □

Lemma 4.7 *$D^2 f(q)$ is given by the matrix $(\partial^2_{k\ell} f(q))_{1 \leq k,\ell \leq n} - d\, f(q)\, I_n$.*

PROOF. Consider $g(t) := f((1+t^2)^{-1/2}(1, tV)) = (1+t^2)^{-d/2} f(1, tV)$ for $V \in \mathbb{R}^n$ of length 1. By definition of the second order derivative we have $D^2 f(q)(V,V) = g''(0)$, cf. (20). A calculation yields $g''(0) = \sum_{k\ell} \partial^2_{k\ell} f(q) V_k V_\ell - d\, f(q)$. □

We show now that the first and second order derivatives of f inherit the invariance property from f.

Lemma 4.8 *Let $f \in H_{d,n}$ be an invariant Gaussian random polynomial. Then $(f(q), Df(q), D^2 f(q)) \in \mathbb{R} \times \mathbb{R}^n \times \Sigma_n$ is $O(n)$-invariant in the sense of Lemma 3.6. In particular, $Df(q)$ is independent of $f(q)$ and $D^2 f(q)$.*

PROOF. Consider for fixed $g \in O(n+1)$ the transformed polynomial $h := gf$, that is, $h(x) = f(g^T x)$. By assumption, h has the same distribution as f. This implies that the random vector $(h, \partial_k h, \partial^2_{k\ell} h)$ is equivalent to $(f, \partial_k f, \partial^2_{k\ell} f)$. (This can be shown by expanding f and g in a fixed basis of $H_{d,n}$ with random real coefficients.) We conclude that $(h(q), Dh(q), D^2 h(q)) = (f(q), gDf(q), gD^2 f(q) g^T)$ is equivalent to $(f(q), Df(q), D^2 f(q))$. □

The following proposition from [32] says that the parameter of $f \in H_{d,n}$ equals the parameter of $D^2 f(q)$, up to a scaling factor. Since the proof in [32] is incomplete, we provide a different proof in the appendix. The assumption of a Gaussian random polynomial is only made for simplifying the statement and could be replaced by suitable regularity conditions.

Proposition 4.9 *Suppose that $f \in H_{d,n}$ is an invariant Gaussian random polynomial. Then:*

(i) $\mathbb{E}(f(q) \operatorname{tr} D^2 f(q)) = -n\delta(f)\, \mathbb{E} f(q)^2$.

(ii) $\delta(D^2 f(q)) = \delta(f)\ \mathbb{E} f(q)^2$.

Corollary 4.10 *Suppose that $f \in H_{d,n}$ is an invariant Gaussian random polynomial. Let W_c be the random matrix $D^2 f(q)$ conditioned on $f(q) = 0$. Then we have $\delta(W_c) = \delta(f)(1 - \delta(f))\, \mathbb{E} f(q)^2$.*

PROOF. Put $u = f(q)$ and $W = D^2 f(q)$. Lemma 4.8 implies that (u, W) is $O(n)$-invariant. Proposition 4.9 yields $\delta(W) = \delta(f)\ \mathbb{E}\,u^2$ and $\mathbb{E}\,(u \operatorname{tr} W) = -n\delta(f)\,\mathbb{E}\,u^2$. The assertion follows now from Corollary 3.7 $\square$

5 Random real projective varieties

We give here the proof of the main Theorem 1.1. Starting from Weyl's tube formula (8) we present in §5.1 a version of a "Rice formula" for curvature coefficients. We then proceed by a careful probabilistic analysis of that formula, making heavily use of the invariance under the orthogonal group. In order to do so, we need all the auxilary material on invariant random vectors, matrices, and polynomials that was collected in §3–§4.

We remark that the kinematic formula of integral geometry (Theorem 2.2) would allow to reduce to the considerably simpler case of one equation. However, in view of a further development of the theory (higher moments), we will not use the kinematic formula here but instead give a fully self-contained probabilistic proof. An alternative proof based on the kinematic formula will be presented in §6.

5.1 A Rice formula for expected curvature coefficients

In a first step we are going to derive a somewhat more explicit form of Weyl's tube formula (8) for the zero set of homogeneous polynomials in S^n.

Let $f_1, \ldots, f_s \in \mathbb{R}[X_0, \ldots, X_n]$ be homogeneous real polynomials of the degrees $d_1, \ldots, d_s$ $(1 \leq s \leq n)$. They define a differentiable map $f\colon S^n \to \mathbb{R}^s$. For a point $x \in S^n$ we denote by $Df(x)\colon T_xS^n \to \mathbb{R}^s$ and $D^2f(x)\colon T_xS^n \times T_xS^n \to \mathbb{R}^s$ the first and second order derivative of f at x.

For the rest of this section we assume that $x \in S^n$, $f(x) = 0$, and $\operatorname{rank} Df(x) = s$. Then, locally at x, the zero set $Z = \mathcal{Z}(f)$ is a smooth Riemannian submanifold of S^n of dimension $n-s$. The kernel of $Df(x)$ equals the tangent space T_xZ of Z at x. We denote the inverse of the restriction of $Df(x)$ to the orthogonal complement $T_xZ^\perp$ of T_xZ by $Df(x)^\dagger\colon \mathbb{R}^s \to T_xZ^\perp$ (Borel-Moore inverse, compare §3.4).

The following, certainly well known lemma, expresses the second fundamental form $\mathrm{II}_Z(x)$ of Z at x (cf. §2.1) in terms of the *Hessian* $Hf(x)\colon T_xZ \times T_xZ \to \mathbb{R}^s$, which we define as the restriction of $D^2f(x)$ to $T_xZ \times T_xZ$. Since we could not find an appropriate reference, we have included a proof in the appendix.

Lemma 5.1 *Under the above assumptions we have for $V, W \in T_xZ$ and $\nu \in (T_xZ)^\perp$*

$$\langle L_Z(x,\nu)(V), W\rangle = \mathrm{II}_Z(x)(V,W,\nu) = -\langle \nu, Df(x)^\dagger Hf(x)(V,W)\rangle.$$

The derivative $Df_\sigma(x)\colon T_xS^n \to \mathbb{R}$ can be identified with a vector in T_xS^n via the inner product on T_xS^n. Suppose $(N_1, \ldots, N_s)$ is the orthonormal basis of $(T_xZ)^\perp$ obtained from $(Df_1(x), \ldots, Df_s(x))$ by Gram-Schmidt orthogonalization. We use the orthogonal map $Qf(x)\colon \mathbb{R}^s \to (T_xZ)^\perp,\ u \mapsto \sum_{\sigma=1}^s u_\sigma N_\sigma$ to describe unit normal vectors in $(T_xZ)^\perp$ by coordinates. We thus define a Weingarten map

$$Lf(x,u) := L_Z(x, Qf(x)(u))$$

of Z at x in direction parameterized by $u \in S^{s-1}$ (compare §2.1). According to Lemma 5.1, the Weingarten map is explicitly characterized by

$$\begin{aligned}\langle Lf(x,u)(V), W\rangle &= -\langle Qf(x)(u), Df(x)^\dagger Hf(x)(V,W)\rangle \\ &= -\langle (Df(x)^\dagger)^T Qf(x)(u), Hf(x)(V,W)\rangle, \end{aligned} \tag{21}$$

where $(Df(x)^\dagger)^T\colon (T_xZ)^\perp \to \mathbb{R}^s$ denotes the adjoint map of $Df(x)^\dagger\colon \mathbb{R}^s \to T_xZ^\perp$.

We suppose now that $\operatorname{rank} Df(x) = s$ for all $x \in Z = \mathcal{Z}(f)$, i.e., the hypersurfaces $\mathcal{Z}(f_i)$ intersect transversally. Then Z is a compact smooth submanifold of S^n of dimension $n-s$. By the transformation theorem, Weyl's formula (8) for the volume of tubes can be rewritten as

$$\operatorname{vol}(T(Z,\alpha)) = \int_{x\in Z}\int_{t\in B_a} \frac{\det(\mathrm{id} - \|t\| Lf(x, t/\|t\|))}{(1+\|t\|^2)^{(n+1)/2}}\, dt\ dZ,$$

where $a = \tan\alpha$ is sufficiently small and $B_a := \{t \in \mathbb{R}^s \mid \|t\| \le a\}$ is the ball of radius a in $\mathbb{R}^s$. Combining this with (3) we obtain

$$\int_{x\in Z}\int_{t\in B_a} \frac{\det(\mathrm{id} - \|t\| Lf(x, t/\|t\|))}{(1+\|t\|^2)^{(n+1)/2}}\, dt\ dZ = \sum_{j=0}^{\lfloor\frac{n-s}{2}\rfloor} K_{s+2j}(Z)\, J_{s+2j}(\alpha). \tag{22}$$

This expansion is valid for all $0<\alpha<\pi/2$ since the functions on both sides of (22) are analytic.

We can now state the announced Rice formula for expected curvatures.

Theorem 5.2 *Suppose that $f = (f_1,\ldots,f_s) \in H_{d_1,n}\times\cdots\times H_{d_s,n}$ is a Gaussian random vector. Hence $f(x)\in\mathbb{R}^s$ is a Gaussian random vector for any $x\in S^n$, and we shall denote its density function by $p_{f(x)}\colon \mathbb{R}^s\to\mathbb{R}$. Let $u\in S^{s-1}$ be a random unit vector uniformly distributed in the sphere and independent of f. Define the function $\psi\colon [0,\infty)\times S^n \to \mathbb{R}$ by the following conditional expectation for $(r,x)\in[0,\infty)\times S^n$*

$$\psi(r,x) := p_{f(x)}(0)\, \mathbb{E}_{f,u}\left[\sqrt{\det(Df(x)Df(x)^T)}\det(\mathrm{id} - rLf(x,u)) \Big/ f(x) = 0\right].$$

By taking a spherical average we define the function

$$\Psi\colon [0,\infty)\to\mathbb{R},\ \Psi(r) := \frac{1}{\mathcal{O}_n}\int_{x\in S^n} \psi(r,x)\, dS^n.$$

Then we have for $0<\alpha<\pi/2$ and $a = \tan\alpha$

$$\sum_{j=0}^{\lfloor\frac{n-s}{2}\rfloor} \mathbb{E} K_{s+2j}(\mathcal{Z}(f))\, J_{s+2j}(\alpha) = \mathcal{O}_n\mathcal{O}_{s-1}\int_0^a \frac{r^{s-1}\Psi(r)}{(1+r^2)^{(n+1)/2}}\, dr.$$

Proof. The proof uses similar ideas as in [1, §5.1] and [3, Theorem 1].

Fix $f_i \in H_{d_i,n}$ for $1 \le i \le s$ and consider the corresponding map $f\colon S^n \to \mathbb{R}^s$. The *normal Jacobian* $NJ(f)(x) := \sqrt{\det(Df(x)Df(x)^T)}$ of f at x has the following geometric meaning. Suppose $x \in S^n$ is a regular point of f and consider the restriction $(\ker Df(x))^\perp \to \mathbb{R}^s$ of $Df(x)$ to the orthogonal complement of $\ker Df(x)$. Then $NJ(f)(x)$ of f equals the absolute value of the determinant of this map, cf. (15).

Recall that $B_a = \{t \in \mathbb{R}^s \mid \|t\| \le a\}$. Consider the following differentiable map of Riemannian manifolds

$$F\colon S^n \times B_a \to \mathbb{R}^s, (x,t) \mapsto f(x).$$

The normal Jacobian of F at $(x,t) \in S^n \times B_a$ satisfies $NJ(F)(x,t) = NJ(f)(x)$.

It is convenient to introduce the following function associated with f

$$g_f\colon S^n \times B_a \to \mathbb{R},\ g_f(x,t) := \frac{\det(\mathrm{id} - \|t\| Lf(x, t/\|t\|))}{(1+\|t\|^2)^{(n+1)/2}}.$$

Note that $\mathrm{vol}(T(Z,\alpha)) = \int_Z \int_{B_a} g_f(x,t)\, dt\, dZ$. Moroever, consider the following integrable function for $\delta > 0$

$$\varphi_\delta\colon S^n \times B_a \to \mathbb{R},\ \varphi(x,t) = \begin{cases} 1 & \text{if } \|f(x)\|_\infty < \delta \\ 0 & \text{otherwise.} \end{cases}$$

The coarea formula (cf. [7, p. 241] or [40, III.§2]) applied to F yields

$$\begin{aligned} &\int_{(x,t)\in S^n\times B_a} \varphi_\delta(x,t) g_f(x,t) NJ(f)(x)\ d(S^n \times B_a) \\ &= \int_{y\in\mathbb{R}^s} \int_{(x,t)\in f^{-1}(y)\times B_a} \varphi_\delta(x,t) g_f(x,t)\ d(f^{-1}(y)\times B_a)\ dy \\ &= \int_{y\in(-\delta,\delta)^s} \int_{(x,t)\in f^{-1}(y)\times B_a} g_f(x,t)\ d(f^{-1}(y)\times B_a)\ dy. \end{aligned}$$

Dividing by $(2\delta)^s$ and taking the expectation over f with respect to the given Gaussian distribution, we obtain

$$\begin{aligned} &\int_{(x,t)\in S^n\times B_a} \frac{1}{(2\delta)^s} \mathbb{E}_f\left(\varphi_\delta(x,t) g_f(x,t) NJ(f)(x)\right)\ d(S^n\times B_a) \\ &= \frac{1}{(2\delta)^s} \int_{y\in(-\delta,\delta)^s} \mathbb{E}_f\Big(\int_{(x,t)\in f^{-1}(y)\times B_a} g_f(x,t)\ d\left(f^{-1}(y)\times B_a\right)\Big)\, dy. \end{aligned} \tag{23}$$

It is not hard to see that the integrand on the right-hand side depends continously on y. Therefore, the right-hand side of (23) converges for $\delta \to 0$ to

$$\mathbb{E}_f\Big(\int_{(x,t)\in f^{-1}(0)\times B_a} g_f(x,t)\ d\left(f^{-1}(0)\times B_a\right)\Big).$$

For fixed $(x,t) \in S^n \times B_a$ the integrand on the left-hand side of (23) can be written as an integral over conditional expectations as follows

$$\frac{1}{(2\delta)^s} \int_{y \in (-\delta,\delta)^s} p_{f(x)}(y)\, \mathbb{E}_f \left(g_f(x,t) NJ(f)(x) \Big/ f(x) = y \right)\, dy. \tag{24}$$

By continuity, the limit of this function for $\delta \to 0$ equals

$$p_{f(x)}(0)\, \mathbb{E}_f \Big(g_f(x,t) NJ(f)(x) \Big/ f(x) = 0 \Big).$$

The integrand of (24) is a continuous function of (x,t,y) and therefore bounded by some constant M on $S^n \times B_a \times \{y \in \mathbb{R}^s \mid \|y\| \leq 1\}$. Hence (24) is as well bounded by M for all $0 < \delta \leq 1$. We may therefore apply Lebesgue's Theorem and interchange in (23) the integral over (x,t) and the limit for $\delta \to 0$ obtaining

$$\begin{aligned} &\int_{(x,t) \in S^n \times B_a} p_{f(x)}(0)\, \mathbb{E}_f \Big(g_f(x,t) NJ(f)(x) \Big/ f(x) = 0 \Big)\, d(S^n \times B_a) \\ &\qquad = \mathbb{E}_f \left(\int_{(x,t) \in f^{-1}(0) \times B_a} g_f(x,t)\, d\left(f^{-1}(0) \times B_a\right) \right). \end{aligned} \tag{25}$$

Note that for $r > 0$ (with uniform random $u \in S^{s-1}$ independent of f)

$$\frac{1}{(1+r^2)^{(n+1)/2}} \psi(r,x) = p_{f(x)}(0)\, \mathbb{E}_{f,u} \Big(g_f(x,ru) NJ(f)(x) \Big/ f(x) = 0 \Big).$$

Hence, using polar coordinates $t = ru$, the left-hand side of (25) can be written as

$$\mathcal{O}_n \mathcal{O}_{s-1} \int_0^a \frac{r^{s-1} \Psi(r)}{(1+r^2)^{(n+1)/2}}\, dr.$$

It remains to interpret the right-hand side of (25). According to (22), we have

$$\int_{(x,t) \in f^{-1}(0) \times B_a} g_f(x,t)\, d\left(f^{-1}(y) \times B_a\right) = \sum_{j=0}^{\lfloor \frac{n-s}{2} \rfloor} K_{s+2j}(Z)\, J_{s+2j}(\alpha)$$

for any $f = (f_1, \ldots, f_s) \in H_{d_1,n} \times \cdots \times H_{d_s,n}$ such that the $\mathcal{Z}(f_i)$ intersect transversally, and for all $0 < \alpha \leq \pi/2$, $a = \tan \alpha$. By Sard's lemma, transversal intersection happens almost surely. Hence, taking expectations, we see that the right-hand side of (25) equals

$$\sum_{j=0}^{\lfloor \frac{n-s}{2} \rfloor} \mathbb{E} K_{s+2j}\left(\mathcal{Z}(f)\right) J_{s+2j}(\alpha),$$

which completes the proof. $\square$

5.2 Expected characteristic polynomial of Weingarten map

In order to prove the main Theorem 1.1 we will evaluate Theorem 5.2 for independent Gaussian polynomials f_σ having invariant distributions. We write $\delta_\sigma = \delta(f_\sigma)$ for the parameter of f_σ. We may assume without loss of generality that $\mathbb{E} f_\sigma(q)^2 = 1$ for all $1 \leq \sigma \leq s$ (scaling does not change the parameter of f_σ). Hence $f_\sigma(q)$ is standard normal and the joint distribution of $f(q)$ has the density $p_{f(q)}(y) = (2\pi)^{-s/2}\exp(-\frac{1}{2}(y_1^2 + \cdots y_s^2))$. In particular, $p_{f(q)}(0) = (2\pi)^{-s/2}$.

We proceed by a sequence of intermediate steps. For $r, u \in [0,\infty) \times S^{s-1}$ and a fixed matrix $M \in \mathbb{R}^{s\times n}$ of rank s we consider the following conditional expectation

$$\mathcal{E}(r,u,M) := \mathbb{E}_f \left[\det(\mathrm{id} - rLf(q,u)) \Big/ f(q) = 0, Df(q) = M\right]. \tag{26}$$

Recall the characterization (21) of $Lf(q,u)$ in terms of $Df(q)$ and $D^2f(q)$. From Lemma 4.8 we know that $D^2f(q)$ is independent of $Df(q)$. Hence the above expectation may be taken with respect to the distribution of $D^2f(q)$ conditioned solely on the event $f(q) = 0$.

Lemma 5.3 *For $r, u \in [0,\infty) \times S^{s-1}$ and a fixed matrix $Df(q)$ of rank s we set*

$$v = (v_1, \ldots, v_s)^T := \mathrm{diag}(\delta_1^{1/2}, \ldots, \delta_s^{1/2})\,(Df(q)^\dagger)^T Qf(q)(u). \tag{27}$$

Then we have (recall (12))

$$\mathcal{E}(r,u,Df(q)) = \sum_{j=0}^{\lfloor\frac{n-s}{2}\rfloor} r^{2j}\gamma_{2j}\binom{n-s}{2j}\left(\sum_{\sigma=1}^{s}(1-\delta_\sigma)v_\sigma^2\right)^j.$$

Proof. Proposition 3.8 is the key to this result. As usual let T_qZ denote the kernel of $Df(q)$ and recall that the Hessian $Hf(q)$ was defined as the restriction of the bilinear map $D^2f(q)$ to $T_qZ \times T_qZ$. For the following introduce an orthonormal basis adapted to $\mathbb{R}^n = T_qZ \oplus T_qT^\perp$ (or observe that §3.2 could have beeen presented in a coordinate-free way).

The matrix of $Lf(q,u)$ is $O(n-s)$-invariant and Gaussian. The same is true for the random matrix $Lf(q,u)_{\mathrm{cond}}$, which is defined as the random matrix $Lf(q,u)$ conditioned on $f(q) = 0$. In order to apply Proposition 3.8 we need to calculate the parameter of $Lf(q,u)_{\mathrm{cond}}$.

We write $Hf(q) = (Hf_1(q), \ldots, Hf_s(q))$. By identifying (bi)linear maps with their matrices we obtain from Equation (21) that

$$Lf(q,u) = -\sum_{\sigma=1}^{s}\frac{v_\sigma}{\sqrt{\delta_\sigma}}\,Hf_\sigma(q),$$

where $v = (v_1, \ldots, v_s)$ is defined as in (27). Hence Lemma 3.5 implies

$$\delta_L := \delta(Lf(q,u)_{\mathrm{cond}}) = \sum_{\sigma=1}^{s} \frac{v_\sigma^2}{\delta_\sigma}\,\delta(Hf_\sigma(q)_{\mathrm{cond}})$$

using obvious notation. Lemma 4.6 tells us that the restriction of f_σ to T_qZ (whose distribution is invariant under the orthogonal group of T_qZ) has the same parameter δ_σ as f_σ. Corollary 4.10 gives that $\delta(Hf_\sigma(q)_{\mathrm{cond}}) = \delta_\sigma(1-\delta_\sigma)$, hence $\delta_L = \sum_\sigma (1-\delta_\sigma)v_\sigma^2$. Proposition 3.8 implies now with $X \sim N(0,1)$

$$\mathcal{E}(r,u,Df(q)) = \mathbb{E}\,(1 + r\sqrt{\delta_L}\,X)^{n-s}.$$

Hence, taking into account that $\gamma_{2j} = \mathbb{E}X^{2j}$, we conclude

$$\mathbb{E}\,(1 + r\sqrt{\delta_L}\,X)^{n-s} = \sum_{j=0}^{\lfloor\frac{n-s}{2}\rfloor} \binom{n-s}{2j} r^{2j}\,\delta_L^j\,\gamma_{2j},$$

which proves the lemma. □

5.3 Proof of main theorem

We prove now the following reformulation of Theorem 1.1 for zero sets in spheres.

Theorem 5.4 *Suppose that $f_\sigma \in H_{d_\sigma,n}$ are independent centered random polynomials with $O(n+1)$-invariant Gaussian distribution of parameter δ_σ. Consider the random zero set $\mathcal{Z}(f) \subseteq S^n$ where $f = (f_1, \ldots, f_s)$. Then the expectation of the curvature coefficient $K_{s+2j}(\mathcal{Z}(f))$ satisfies*

$$\frac{\mathbb{E}\,K_{s+2j}(\mathcal{Z}(f))}{\mathcal{O}_{n-s-2j}\,\mathcal{O}_{s+2j-1}} = (\delta_1\cdots\delta_s)^{1/2} \sum_{\nu\in\mathbb{N}^s,\,|\nu|=j} (1-\delta_1)^{\nu_1}\cdots(1-\delta_s)^{\nu_s} C_{\nu_1}^{(1)}\cdots C_{\nu_s}^{(1)}$$

for $0 \le j \le \lfloor\frac{n-s}{2}\rfloor$, where the coefficients $C_k^{(1)}$ can be characterized by the generating function

$$(1-Y)^{-1/2} = \sum_{k=0}^{\infty} C_k^{(1)} Y^k = 1 + \frac{1}{2}Y + \frac{3}{8}Y^2 + \frac{5}{16}Y^3 + \cdots. \tag{28}$$

We have $C_0^{(1)} = 1$ and $C_k^{(1)} = \frac{1\cdot 3\cdot 5\cdots(2k-1)}{k!\,2^k} = \frac{(2k)!}{4^k\,k!^2}$ for $k > 0$.

In the case where all $\delta_\sigma := \delta$ are equal the result simplifies to

$$\mathbb{E}\,K_{s+2j}(\mathcal{Z}(f)) = \delta^{s/2}\,(1-\delta)^j\,\mathcal{O}_{n-s-2j}\mathcal{O}_{s+2j-1}C_j^{(s)} \quad \text{for } 0 \le j \le \lfloor\tfrac{n-s}{2}\rfloor,$$

where the $C_j^{(s)}$ are characterized as the coefficients of the power series

$$(1-Y)^{-s/2} = \sum_{k=0}^{\infty} C_k^{(s)} Y^k.$$

More specifically, we have $C_0^{(s)} = 1$ and for $k > 0$

$$C_k^{(s)} = \frac{s(s+2)(s+4)\cdots(s+2k-2)}{k!\,2^k}.$$

Proof of Theorem 5.4. Put $D := \operatorname{diag}(\sqrt{\delta_1}, \ldots, \sqrt{\delta_s})$. Since $\delta_\sigma^{-1/2} Df_\sigma(q)$ is standard normal distributed in $\mathbb{R}^n$ and the f_σ are independent, we can write $Df(q) = DA$, where $A \in \mathbb{R}^{s\times n}$ is a random matrix with independent standard Gaussian entries. Note that $\det(Df(q)Df(q)^T) = \delta_1 \cdots \delta_s \det(AA^T)$ and $Df(q)^\dagger = A^\dagger D^{-1}$.

Let $u \in S^{s-1}$ be a uniformly distributed random unit vector which is independent of A. The function ψ introduced in Theorem 5.2 satisfies for $r > 0$

$$\psi(r,q) = (2\pi)^{-s/2}(\delta_1\cdots\delta_s)^{1/2}\,\mathbb{E}_{A,u}\left[\sqrt{\det AA^T}\;\mathcal{E}(r,u,DA)\right].$$

Lemma 5.3 tells us that

$$\mathcal{E}(r,u,DA)) = \sum_{j=0}^{\lfloor\frac{n-s}{2}\rfloor} r^{2j}\gamma_{2j}\binom{n-s}{2j}\left(\sum_{\sigma=1}^{s}(1-\delta_\sigma)v_\sigma^2\right)^j,$$

where $v \in \mathbb{R}^s$ is the image of u under the linear endomorphism

$$D(Df(q)^\dagger)^T Qf(q)(u) = (A^\dagger)^T Q_A(u)$$

of $\mathbb{R}^s$ (recall the definition of Q_A in (16)). We make the multinomial expansion

$$\left(\sum_{\sigma=1}^{s}(1-\delta_\sigma)v_s^2\right)^j = \sum_{\nu\in\mathbb{N}^s,\,|\nu|=j}\binom{j}{\nu}(1-\delta_1)^{\nu_1}\cdots(1-\delta_s)^{\nu_s}v_1^{2\nu_1}\cdots v_s^{2\nu_s}.$$

Thus we need to compute for $\nu \in \mathbb{N}^s$ with $|\nu| = j$

$$\mathbb{E}_{A,u}\left[\sqrt{\det AA^T}\;v_1^{2\nu_1}\cdots v_s^{2\nu_s}\right].$$

Proposition 3.12 determines the joint distribution of $(\sqrt{\det AA^T}, v)$ for random A and u. Accordingly, we write with a uniformly distributed $w \in S^{s-1}$ that is independent of A:

$$\mathbb{E}_{A,u}\left[\sqrt{\det AA^T}\;v_1^{2\nu_1}\cdots v_s^{2\nu_s}\right] = \mathbb{E}_{A,u}\left[\operatorname{vol}(A_1,\ldots,A_{s-1})\,\frac{1}{\|A_s^\perp\|^{2j-1}}w_1^{2\nu_1}\cdots w_s^{2\nu_s}\right].$$

It is well known that [42]

$$\mathbb{E}_{w\in S^{s-1}}\left[w_1^{2\nu_1}\cdots w_s^{2\nu_s}\right]=\frac{\gamma_{2\nu_1}\cdots\gamma_{2\nu_s}}{s(s+2)\cdots(s+2j-2)}.$$

By using Lemma 3.11 we obtain

$$\mathbb{E}_{A,u}\left[\sqrt{\det AA^T}\ v_1^{2\nu_1}\cdots v_s^{2\nu_s}\right]=\frac{\mathcal{O}_{n-s}}{2(2\pi)^{\frac{n-s}{2}}}\cdot\frac{\gamma_n\gamma_{n-s+1-2j}}{\gamma_{n-s+1}}\,\frac{\gamma_{2\nu_1}\cdots\gamma_{2\nu_s}}{s(s+2)\cdots(s+2j-2)}.$$

This formula can be considerably simplified. We put $C_k^{(1)}:=\frac{(2k)!}{4^k\,k!^2}$ for $k\in\mathbb{N}$.

Claim. We have

$$\frac{\mathcal{O}_n\mathcal{O}_{s-1}}{(2\pi)^{s/2}\mathcal{O}_{n-s-2j}\mathcal{O}_{s+2j-1}}\gamma_{2j}\binom{n-s}{2j}\binom{j}{\nu}\mathbb{E}_A\left[\sqrt{\det AA^T}\ v_1^{2\nu_1}\cdots v_s^{2\nu_s}\right]=C_{\nu_1}^{(1)}\cdots C_{\nu_s}^{(1)}.$$

In order to verify this recall first that

$$\mathcal{O}_n=\frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})},\quad \gamma_n=\frac{1}{\sqrt{\pi}}\,2^{n/2}\,\Gamma(\frac{n+1}{2}).$$

From $\Gamma(x+1)=x\Gamma(x)$ and $\Gamma(1/2)=\sqrt{\pi}$ we get

$$\Gamma(m+1)=m!,\quad \Gamma(m+1/2)=(m-1/2)(m-3/2)\cdots 1/2\sqrt{\pi}.$$

Using the above it is straightforward to check that

$$\mathcal{O}_n\,\gamma_n=2(2\pi)^{n/2},\quad \frac{n!\,\mathcal{O}_n}{\gamma_{n+1}}=(2\pi)^{\frac{n+1}{2}},\quad (2\pi)^j\,\frac{\mathcal{O}_{s-1}}{\mathcal{O}_{s-1+2j}}=s(s+2)\cdots(s+2j-2).$$

Moreover, recall from (13) that $(2j)!=2^j\,j!\gamma_{2j}$ and note that $C_k^{(1)}=\frac{\gamma_{2k}}{2^k\,k!}$. The claim follows by simplifying the formula using the above stated equations in a straightforward (but tedious) way.

Combining what we have shown so far we obtain

$$\frac{\mathcal{O}_n\mathcal{O}_{s-1}\,\psi(r,q)}{\mathcal{O}_{n-s-2j}\mathcal{O}_{s+2j-1}}=(\delta_1\cdots\delta_s)^{1/2}\sum_{|\nu|\le\lfloor\frac{n-s}{2}\rfloor}(1-\delta_1)^{\nu_1}\cdots(1-\delta_s)^{\nu_s}C_{\nu_1}^{(1)}\cdots C_{\nu_s}^{(1)}r^{2|\nu|}.$$

We apply now Theorem 5.2. From the $O(n+1)$-invariance it follows that $\psi(r,x)=\psi(r,q)$ for all $x\in S^n$, hence $\Psi(r)=\psi(r,q)$. Let $0<\alpha<\pi/2$ and put $a=\tan\alpha$. By substituting $r=\tan\rho$ we obtain

$$\int_0^a\frac{r^{s+2j-1}}{(1+r^2)^{(n+1)/2}}\,dr=J_{s+2j}(\alpha).$$

We conclude with Theorem 5.2 that

$$\begin{aligned}
&\sum_{j=0}^{\lfloor\frac{n-s}{2}\rfloor} \frac{\mathbb{E}\,K_{s+2j}(\mathcal{Z}(f))}{\mathcal{O}_{n-s-2j}\mathcal{O}_{s+2j-1}}\,J_{s+2j}(\alpha) \\
&\quad = \frac{\mathcal{O}_n\mathcal{O}_{s-1}}{\mathcal{O}_{n-s-2j}\mathcal{O}_{s+2j-1}}\int_0^a \frac{r^{s-1}\,\Psi(r)}{(1+r^2)^{(n+1)/2}}\,dr \\
&\quad = (\delta_1\cdots\delta_s)^{1/2}\sum_{|\nu|\le\lfloor\frac{n-s}{2}\rfloor}(1-\delta_1)^{\nu_1}\cdots(1-\delta_s)^{\nu_s}C_{\nu_1}^{(1)}\cdots C_{\nu_s}^{(1)}\,J_{s+2|\nu|}(\alpha).
\end{aligned}$$

By comparing the coefficients of the linearly independent functions $J_k(\alpha)$, the stated formula for $\mathbb{E}\,K_{s+2j}(\mathcal{Z}(f))$ follows.

To settle the case where all δ_σ are equal just note $(1-Y)^{-1/2}=\sum_{k=0}^{\infty}C_k^{(1)}Y^k$ and $(1-Y)^{-s/2}=\sum_{k=0}^{\infty}C_k^{(s)}Y^k$ implies that $C_j^{(s)}=\sum_{|\nu|=j}C_{\nu_1}^{(1)}\cdots C_{\nu_s}^{(1)}$. □

6 Alternative proof of the main result

We show here that Theorem 1.1 can be quickly derived from the knowledge of the expected Euler characteristic of a random projective hypersurface $\mathcal{Z}(f)$ for an invariant centered Gaussian random polynomial f. The key of this reduction is the kinematic formula and the generalized Gauss-Bonnet theorem.

In this section, $\mathcal{Z}(f)$ stands for the zero set in $\mathbb{P}^n$. Suppose $f\in H_{d,2\ell+1}$ has invariant centered Gaussian distribution with parameter δ. We define $\chi_\ell(\delta) := \mathbb{E}\chi(\mathcal{Z}(f))$. Theorem 1.1 in the case of one equation ($s=1$) yields

$$\chi_\ell(\delta)=\delta^{1/2}\sum_{k=0}^{\ell}C_k^{(1)}(1-\delta)^k. \tag{29}$$

Taking into account Equation (28), we get a closed form expression for the generating function $\chi(\delta;T)$ of $\chi_\ell(\delta)$ as follows:

$$\begin{aligned}
\chi(\delta;T) &:= \sum_{\ell=0}^{\infty}\chi_\ell(\delta)\,T^{2\ell}=\delta^{1/2}\sum_{k=0}^{\infty}C_k^{(1)}(1-\delta)^kT^{2k}\sum_{\ell=k}^{\infty}T^{2(\ell-k)} \\
&= \frac{\delta^{1/2}}{(1-T^2)(1-(1-\delta)T^2)^{1/2}}. \qquad (30)
\end{aligned}$$

This formula can also be readily deduced from Podkorytov's result [32], cf. (2).

Using the kinematic formula we can prove a stability result for $\mathbb{E}\mu_e(\mathcal{Z}(f))$.

Lemma 6.1 *Suppose $f\in H_{d,n}$ is $O(n+1)$-invariant and $n'\le n$. Then the restriction $f'\in H_{d,n'}$ of f to $\mathbb{R}^{n'+1}$ is $O(n'+1)$-invariant and has the same parameter. For $0\le e<n'$, e even, we have $\mathbb{E}\mu_e(\mathcal{Z}(f'))=\mathbb{E}\mu_e(\mathcal{Z}(f))$.*

Proof. The first part of the statement was already established in Lemma 4.6. By Theorem 2.2 (in the version for $\mathbb{P}^n$) we have for almost all $f \in H_{d,n}$, as $\mu(\mathbb{P}^{n'}; T) = 1$,

$$\int \mu(\mathcal{Z}(f) \cap g\mathbb{P}^{n'}; T)\, dg \equiv \mu(\mathcal{Z}(f); T) \bmod T^{n'},$$

where the integral is with respect to the Haar measure of $O(n+1)$ scaled such that the volume of $O(n+1)$ equals 1. Taking the expectation over f and interchanging with the integral over g we obtain

$$\int \mathbb{E}_f\,(\mu(\mathcal{Z}(f) \cap g\mathbb{P}^{n'}; T))\, dg \equiv \mathbb{E}_f\ \mu(\mathcal{Z}(f); T) \bmod T^{n'}.$$

By the invariance of the distribution of f, the integrand is independent of g, hence the integral equals $\mathbb{E}_f\,(\mu(\mathcal{Z}(f) \cap \mathbb{P}^{n'}; T))$. This expectation equals $\mathbb{E}_{f'}(\mu(\mathcal{Z}(f'); T))$, which finishes the proof. □

Lemma 6.2 *Suppose that $f \in H_{d,n}$ is $O(n+1)$-invariant with parameter δ. Then we have $\mathbb{E}\mu_0(\mathcal{Z}(f)) = \chi_0(r)$ and for $1 \le k \le (n-1)/2$*

$$\mathbb{E}\,\mu_{2k}(\mathcal{Z}(f)) = \chi_k(\delta) - \chi_{k-1}(\delta).$$

In particular, $\mathbb{E}\,\mu_{2k}(\mathcal{Z}(f))$ depends only on k and δ and not on the dimension n of the ambient space.

Proof. We denote by f' and f'' the restrictions of f to $\mathbb{R}^{2k+2}$ and $\mathbb{R}^{2k}$, respectively. Theorem 2.1 (in the version for $\mathbb{P}^n$) implies that

$$\chi_k(\delta) = \sum_{i=0}^{k} \mathbb{E}\,\mu_{2i}(\mathcal{Z}(f')), \quad \chi_{k-1}(\delta) = \sum_{i=0}^{k-1} \mathbb{E}\,\mu_{2i}(\mathcal{Z}(f'')).$$

By Lemma 6.1 we have $\mathbb{E}\,\mu_{2i}(\mathcal{Z}(f)) = \mathbb{E}\,\mu_{2i}(\mathcal{Z}(f')) = \mathbb{E}\,\mu_{2i}(\mathcal{Z}(f''))$ for $0 \le i \le k-1$ and $\mathbb{E}\mu_{2k}(\mathcal{Z}(f)) = \mathbb{E}\mu_{2k}(\mathcal{Z}(f'))$. Subtracting the above two equations, we obtain $\chi_k(\delta) - \chi_{k-1}(\delta) = \mathbb{E}\,\mu_{2k}(\mathcal{Z}(f))$ as claimed. The assertion for $k=0$ is obvious. □

We proceed now with an alternative proof of Theorem 1.1

Proof of Theorem 1.1. By Lemma 6.2 the formal power series

$$\mu(\delta; T) := \sum_{k=0}^{\infty} \mathbb{E}\,\mu_{2k}(\mathcal{Z}(f))\, T^{2k}$$

satisfies $\mu(\delta; T) = (1 - T^2)\,\chi(\delta; T)$. Equation (30) implies that

$$\mathbb{E}\mu(\delta; T) = \delta^{1/2}(1 - (1-\delta)T^2)^{-1/2}. \tag{31}$$

Suppose now that $f_\sigma \in H_{d_\sigma,n}$ are independent random variables with invariant centered Gaussian distribution of parameter δ_σ for $1 \le \sigma \le s \le n$. Write $\underline{f} := (f_1,\ldots,f_{s-1})$. The kinematic formula (Theorem 2.2 in the version for $\mathbb{P}^n$) tells us that for almost all $\underline{f}, f_s$

$$\int \mu(\mathcal{Z}(\underline{f}) \cap g\mathcal{Z}(f_s);T)\,dg \equiv \mu(\mathcal{Z}(\underline{f});T)\ \mu(\mathcal{Z}(f_s);T) \bmod T^{n-s+1},$$

where the integral is over $O(n+1)$ with respect to the Haar measure scaled to 1. We take the expectation with respect to $\underline{f}$ and f_s. Taking their independence into account we get

$$\int \mathbb{E}_{\underline{f},f_s}\,\mu(\mathcal{Z}(\underline{f}) \cap g\mathcal{Z}(f_s);T)\,dg \equiv \mathbb{E}\mu(\mathcal{Z}(\underline{f});T)\ \mathbb{E}\mu(\mathcal{Z}(f_s);T) \bmod T^{n-s+1}.$$

By invariance, the integrand does not depend on g and equals $\mathbb{E}\mu(\mathcal{Z}(f_1,\ldots,f_s);T)$. We conclude by induction that

$$\mathbb{E}\mu(\mathcal{Z}(f_1,\ldots,f_s);T) \equiv \mathbb{E}\mu(\mathcal{Z}(f_1);T)\cdots\mathbb{E}\mu(\mathcal{Z}(f_s);T) \bmod T^{n-s+1}.$$

Plugging in the explicit expression (31) for $\mathbb{E}\mu(\mathcal{Z}(f_i);T)$, the desired formula for the expectation of the curvature polynomial follows. Finally note that $(1-Y)^{-s/2} = \sum_{k=0}^\infty C_k^{(s)} Y^k$. □

Remark 6.3 The above reduction to the computation of the expected Euler characteristic works for *any* $O(n+1)$-invariant distribution of random polynomials. The Gaussian assumption is not needed for the reduction.

Remark 6.4 We briefly outline how (30) can be derived from a general result of Taylor and Adler [41]. Suppose that $f \in H_{d,n}$ is $O(n+1)$-invariant with parameter δ and suppose w.l.o.g. that $\mathbb{E} f(p)^2 = 1$ for $p \in S^n$. We consider f as a centered unit variance Gaussian field on the sphere S^n. Then f defines a Riemannian metric on S^n by $g_p(X_p,Y_p) := \mathbb{E}(X_pf \cdot Y_pf)$ for tangent vectors $X_p, Y_p \in T_pS^n$. It easily follows that $g_p(X_p,Y_p) = \delta\,\langle X_p, Y_p\rangle$, where $\langle\ ,\ \rangle$ denotes the scalar product on T_pS^n. Hence, with respect to this metric, S^n is isometric to the sphere M in $\mathbb{R}^{n+1}$ of radius $R = \sqrt{\delta}$. Theorem 4.1 of [41] states that

$$\mathbb{E}\big(\chi(S^n \cap f^{-1}[u,\infty)\big) = \sum_{j=0}^n \mathcal{L}_j \rho_j(u),$$

with "Lipschitz-Killing curvatures" $\mathcal{L}_j$ of M and functions $\rho_j(u)$ related to Hermite polynomials. Almost surely, $S^n \cap f^{-1}[u,\infty)$ is a compact domain with smooth boundary $S^n \cap f^{-1}(0)$, which implies $\chi(S^n \cap f^{-1}(0)) = 2\chi(S^n \cap f^{-1}[u,\infty))$ if n is odd. The Lipschitz-Killing curvatures of M can be defined via Weyl's formula for the volume of the tubes around M in $\mathbb{R}^{n+1}$

$$\mathrm{vol}(T(M,r)) = \sum_{j=0}^{n} \mathcal{L}_j\, \omega_{n+1-j}\, r^{n+1-j}$$

with $\omega_0 := 1$ and $\omega_k = \mathcal{O}_{k-1}/k$ for $k > 0$. A straightforward calculation yields $\mathcal{L}_j = \frac{2\omega_{n+1}}{\omega_{n+1-j}} \binom{n+1}{j} R^j$ if $j \equiv n \bmod 2$ and $\mathcal{L}_j = 0$ otherwise. Hence we obtain for odd n

$$\mathbb{E}\chi(\mathcal{Z}_{\mathbb{P}^n}(f)) = \frac{1}{2}\mathbb{E}\chi(S^n \cap f^{-1}(0)) = \sum_{1\le j\le n, j \text{ odd}} \frac{2\omega_{n+1}}{\omega_{n+1-j}} \binom{n+1}{j} \delta^{j/2}\rho_j(0).$$

It is possible to derive Equation (29) from this, but we omit the details.

Appendix

Proof of Theorem 2.1. W.l.o.g. $m < n$. We suppose first that n is odd. The submanifold M is a deformation retract of the tube $T_\alpha := T(M,\alpha)$ for sufficiently small $\alpha > 0$, hence $\chi(M) = \lim_{\alpha\to 0}\chi(T_\alpha)$. The generalized Gauss-Bonnet formula applied to the domain T_α in S^n says that (cf. [35, (17.22), p. 303])

$$\frac{1}{2}\mathcal{O}_n\,\chi(T_\alpha) = \sum_{0\le i\le n-1,\, i \text{ even}} \frac{\mathcal{O}_n}{\mathcal{O}_{n-1-i}\mathcal{O}_i} \int_{\partial T_\alpha} \sigma_i(\kappa_1,\ldots,\kappa_{n-1})\,d(\partial T_\alpha).$$

Going to the limit $\alpha \to 0$ and applying Equation (10) we get ($\mu = n - m$ is odd)

$$\begin{aligned}\frac{1}{2}\chi(M) &= \sum_{0\le i\le n-1,\, i \text{ even}} \frac{1}{\mathcal{O}_{n-1-i}\mathcal{O}_i} K_{i+1}(M) \quad (\text{put } i+1 = \mu + e)\\ &= \sum_{0\le e\le m,\, e \text{ even}} \frac{1}{\mathcal{O}_{m-e}\mathcal{O}_{\mu+e-1}} K_{\mu+e}(M) = \sum_{0\le e\le m,\, e \text{ even}} \mu_e(M).\end{aligned}$$

In the case where n is even we argue similarly. The generalized Gauss-Bonnet formula (cf. [35, (17.21), p. 303] applied to T_α says that

$$\frac{1}{2}\mathcal{O}_n\chi(T_\alpha) = \mathrm{vol}(T_\alpha) + \sum_{1\le i\le n-1,\, i \text{ odd}} \frac{\mathcal{O}_n}{\mathcal{O}_{n-1-i}\mathcal{O}_i} \int_{\partial T_\alpha} \sigma_i(\kappa_1,\ldots,\kappa_{n-1})\,d(\partial T_\alpha).$$

Hence, by taking the limit $\alpha \to 0$ and using (10), we get (note that μ is even)

$$\begin{aligned}\frac{1}{2}\chi(M) &= \sum_{1\le i\le n-1,\, i \text{ odd}} \frac{1}{\mathcal{O}_{n-1-i}\mathcal{O}_i} K_{i+1}(M)\\ &= \sum_{0\le e\le m,\, e \text{ even}} \frac{1}{\mathcal{O}_{m-e}\mathcal{O}_{\mu+e-1}} K_{\mu+e}(M) = \sum_{0\le e\le m,\, e \text{ even}} \mu_e(M),\end{aligned}$$

which finishes the proof. $\square$

PROOF OF PROPOSITION 4.9. The covariance function $r(x,y) := \mathbb{E}(f(x)f(y))$ is a polynomial function that is homogeneous of degree d in both sets of variables $x \in \mathbb{R}^{n+1}$ and $y \in \mathbb{R}^{n+1}$. Since the distribution of f is invariant under the action of the orthogonal group, we have $r(gx,gy) = r(x,y)$ for all $g \in O(n+1)$ and $x, y \in \mathbb{R}^{n+1}$. It follows from invariant theory that $r(x,y)$ is a real polynomial in $\|x\|^2, \|y\|^2, \langle x, y\rangle$ (e.g., see [39]). From the fact that r is bihomogeneous of degree (d,d) it is easy to conclude that r has the following form

$$r(x,y) = \sum_{k=0}^{\lfloor d/2 \rfloor} \beta_k \, \|x\|^{2k} \|y\|^{2k} \langle x, y\rangle^{d-2k} \qquad (\beta_k \in \mathbb{R}).$$

We can express the moments of the partial derivatives of f at q by partial derivatives of the covariance function r at (q,q): we have for $1 \le i,j,k,\ell \le n$

$$\begin{aligned} \mathbb{E}\,(\partial^2_{x_i x_j} f(q)\, f(q)) = \partial^2_{x_i x_j} r(q,q),\ \mathbb{E}\,(\partial_{x_i} f(q)\, \partial_{y_k} f(q)) = \partial^2_{x_i y_k} r(q,q), \\ \mathbb{E}\,(\partial^2_{x_i x_j} f(q)\, \partial^2_{y_k y_\ell} f(q)) = \partial^4_{x_i x_j y_k y_\ell} r(q,q). \end{aligned} \tag{32}$$

In order to show the second claim it is convenient to use the abbreviation $a_{ij} := \partial^2_{x_i x_j} f(q)$, $A := (a_{ij})_{1 \le i,j \le n}$. By Lemma (4.7) we have $W := D^2 f(q) = A - df(q) I_n$. We obtain for the parameter of W

$$\begin{aligned} \delta(W) = {} & \frac{1}{n(n-1)} \Big(\mathbb{E}\,(\mathrm{tr} W)^2 - \mathbb{E}\,\|W\|_F^2 \Big) \\ = {} & \frac{1}{n(n-1)} \Big(\mathbb{E}\,(\mathrm{tr} A)^2 - \mathbb{E}\|A\|_F^2 - 2(n-1)d\,\mathbb{E}\,(f(q)\mathrm{tr} A) + n(n-1)d^2 \mathbb{E}\, f(q)^2 \Big) \\ = {} & \frac{1}{n(n-1)} \sum_{i \ne j} \mathbb{E}\,(a_{ii} a_{jj} - a_{ij}^2) - \frac{2}{n} d\, \mathbb{E}\,(f(q)\,\mathrm{tr} A) + d^2 \mathbb{E}\, f(q)^2 \\ = {} & \frac{1}{n(n-1)} \sum_{i \ne j} \Big(\partial^4_{x_i x_i y_j y_j} r - \partial^4_{x_i x_j y_i y_j} r \Big)(q,q) - \frac{2}{n} d \sum_i \partial^2_{x_i x_i} r(q,q) + d^2 r(q,q). \end{aligned}$$

On the other hand,

$$\delta(f)\mathbb{E}\, f(q)^2 = \frac{1}{n} \sum_i \mathbb{E}\,\left(\partial_{x_i} f(q)\right)^2 = \frac{1}{n} \sum_i \partial^2_{x_i y_i} r(q,q).$$

In order to prove the second claim, it suffices to check equality of the above two expressions for $r_k(x,y) = \|x\|^{2k}\|y\|^{2k}\langle x,y\rangle^{d-2k}$ since both expressions are linear in r.

A tedious but straightforward calculation yields for $1 \le i,j \le n$, $i \ne j$,

$$\begin{aligned} r_k(q,q) = 1,\ \partial_{x_i x_i} r_k(q,q) = 2k,\ \partial_{x_i y_i} r_k(q,q) = d - 2k, \\ \partial^4_{x_i x_i y_j y_j} r(q,q) = 4k^2,\ \partial^4_{x_i x_j y_i y_j} r(q,q) = (d-2k)(d-2k-1). \end{aligned}$$

Plugging in this in the above expressions we see that indeed $\delta(W) = \delta(f)\mathbb{E}\, f(q)^2$. The verification of the first claim is similar and a bit simpler. $\square$

Proof of Lemma 5.1. By invariance of the assertion under the orthogonal group it is sufficient to verify the claim at the point $q := (1, 0 \ldots, 0)$ and we may also assume that $T_qZ = \ker Df(q) = \mathbb{R}^{n-s} \times 0^s$. Hence $\partial_{n-s+\tau} f_\sigma(q) = 0$ for $1 \leq \sigma, \tau \leq s$.

Our assumption $\operatorname{rank} Df(q) = s$ allows to apply the implicit function theorem. There is an open subset $U \subseteq \mathbb{R}^{n-s}$ containing the origin and a differentiable map $h: \mathbb{R}^{n-s} \supseteq U \to \mathbb{R}^s$ such that $h(0) = 0$ and

$$\varphi: \mathbb{R}^{n-s} \supseteq U \to S^n, \; u \mapsto \frac{(1, u, h(u))}{\sqrt{1 + \|u\|^2 + \|h(u)\|^2}}$$

is a local diffeomorphism of U onto an open neighborhood of q in Z.

We make the following useful convention on indices: α, β, γ run in the range $1, 2, \ldots, n-s$ while σ, τ, ρ run in the range $1, 2, \ldots, s$. A straightforward calculation shows that

$$\partial_\alpha \varphi_0(0) = 0, \quad \partial_\alpha \varphi_\gamma(0) = \delta_{\alpha\gamma}, \quad \partial_\alpha \varphi_{n-s+\sigma}(0) = \partial_\alpha h_\sigma(0) = 0,$$

where the last equality follows from our assumption $T_qZ = \mathbb{R}^{n-s} \times 0^s$. A similar calculation yields

$$\partial^2_{\alpha,\beta} \varphi_{n-s+\sigma}(0) = \partial^2_{\alpha,\beta} h_\sigma(0).$$

By the definition (6) of the second fundamental form we obtain for $V, W \in T_qZ = \mathbb{R}^{n-s} \times 0^s$ and $\nu \in T_xZ^\perp = 0^{n-s} \times \mathbb{R}^s$

$$\mathrm{II}_Z(x)(V, W, \nu) = \sum_{\alpha,\beta,\sigma} \nu_\sigma \partial^2_{\alpha,\beta} h_\sigma(0) V_\alpha W_\beta. \tag{33}$$

It remains to express $\partial^2_{\alpha,\beta} h_\sigma(0)$ by partial derivatives of f.

By taking the derivative of $f_\rho(1, u, h(u)) = 0$ with respect to u_α we obtain

$$\partial_{X_\alpha} f_\rho(1, u, h(u)) + \sum_\tau \partial_{X_{n-s+\tau}} f_\rho(1, u, h(u)) \cdot \partial_\alpha h_\tau(u) = 0.$$

Differentiating this with respect to u_β and taking into account that $\partial_\alpha h_\tau(0) = 0$ we obtain after a short calculation at $v = 0$

$$\partial^2_{X_\alpha, X_\beta} f_\rho(q) + \sum_\tau \partial_{X_{n-s+\tau}} f_\rho(q) \cdot \partial^2_{\alpha,\beta} h_\tau(0) = 0.$$

We may write this as $\partial^2_{\alpha,\beta} h_\sigma(0) = -\sum_\rho m_{\sigma,\rho} \partial^2_{X_\alpha,X_\beta} f_\rho(q)$, where $(m_{\sigma,\rho})$ denotes the matrix of $Df(q)^\dagger$. Plugging this into (33) we get

$$\begin{aligned} \mathrm{II}_Z(x)(V, W, \nu) &= -\sum_\sigma \nu_\sigma \sum_\rho m_{\sigma,\rho} \sum_{\alpha,\beta} \partial^2_{X_\alpha,X_\beta} f_\rho(q) V_\alpha W_\beta \\ &= \langle \nu, Df(q)^\dagger Hf(q)(V, W) \rangle, \end{aligned}$$

which was to be shown. $\square$

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