

Geometric Complexity Theory and Tensor Rank

[Extended Abstract] *

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ABSTRACT

Mulmuley and Sohoni [25, 26] proposed to view the permanent versus determinant problem as a specific orbit closure problem and to attack it by methods from geometric invariant and representation theory. We adopt these ideas towards the goal of showing lower bounds on the border rank of specific tensors, in particular for matrix multiplication. We thus study specific orbit closure problems for the group $G = GL(W_1) \times GL(W_2) \times GL(W_3)$ acting on the tensor product $W = W_1 \otimes W_2 \otimes W_3$ of complex finite dimensional vector spaces. Let $G_s = SL(W_1) \times SL(W_2) \times SL(W_3)$. A key idea from [26] is that the irreducible G_s -representations occurring in the coordinate ring of the G -orbit closure of a stable tensor $w \in W$ are exactly those having a nonzero invariant with respect to the stabilizer group of w .

However, we prove that by considering G_s -representations, only trivial lower bounds on border rank can be shown. It is thus necessary to study G -representations, which leads to geometric extension problems that are beyond the scope of the subgroup restriction problems emphasized in [25, 26] and its follow up papers. We prove a very modest lower bound on the border rank of matrix multiplication tensors using G -representations. This shows at least that the barrier for G_s -representations can be overcome. To advance, we suggest the coarser approach to replace the semigroup of representations of a tensor by its moment polytope. We prove first results towards determining the moment polytopes of matrix multiplication and unit tensors.

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1. INTRODUCTION

Mulmuley and Sohoni [25, 26] proposed to view the permanent versus determinant problem as a specific orbit closure problem and to attack it by methods from geometric invariant and representation theory. So far there has been little progress with this approach, mainly due to the difficulty of the various arising mathematical problems [6]. It is the goal of this paper to examine and to further develop the collection of ideas from [25, 26] at a problem simpler than the permanent versus determinant, but still of considerable interest for complexity theory.

The complexity of matrix multiplication is captured by the rank of the matrix multiplication tensor, a quantity that, despite intense research efforts, is little understood. Strassen [34] already observed that the closely related notion of border rank has a natural formulation as a specific orbit closure problem. Moreover, it is remarkable that the best known lower bound on the rank of matrix multiplication (Bläser [2]) owes its existence to an explicit construction of an invariant polynomial in the vanishing ideal of certain secant varieties (Strassen [33]).

We carried out the program in [25, 26] for the matrix multiplication versus unit tensor problem. More specifically, we determined the stabilizers (symmetry groups) of the corresponding tensors and verified that they are stable. Moreover, we found explicit representation theoretic characterizations of the irreducible G_s -representations occurring in the coordinate rings of the G -orbit closures of these tensors in terms of nonvanishing of Kronecker coefficients and related quantities (G and G_s stand for a product of general linear groups and special linear groups, respectively, cf. (2.1) and §3.6).

Unfortunately, it turns out that using G_s -representations, only trivial lower bounds on border rank can be shown (Theorem 4.6)! This insight is one of our main results. It does not kill the overall program, but implies that the finer G -representations have to be considered instead. As a consequence, the stability property is not enough to overcome the issue of orbit closures and additional properties, beyond the subgroup restriction problems emphasized in [25, 26], need to be studied. What we have to face is the problem of extending (highest weight) regular functions from an orbit to its orbit closure. It turns out that this can be captured by a single integer k that seems of a geometric nature (cf. Theorem 6.2). Currently we understand the extension problem very little.

In §8 we prove, for the first time, a lower bound on the border rank of matrix multiplication tensors using G -representations. While this bound is still very modest, it shows at least that the barrier for G_s -representations from Theorem 4.6 can be overcome.

A natural approach to advance is to take a coarser, asymptotic viewpoint which replaces the semigroup of representations by their moment polytopes [3, 35]. We prove first results towards determining the moment polytopes of matrix multiplication and unit tensors. This is based on the asymptotic properties of the Kronecker polytope derived in [4].

Due to lack of space some of the proofs had to be omitted in this extended abstract.

2. PRELIMINARIES

2.1 Tensor rank

Let W_1, W_2, W_3 be finite dimensional complex vector spaces of dimensions m_1, m_2, m_3 , respectively. We put $W := W_1 \otimes W_2 \otimes W_3$ and call $\underline{m} = (m_1, m_2, m_3)$ the *format* of W . The elements $w \in W$ shall be called *tensors* and w is called indecomposable if it has the form $w = w_1 \otimes w_2 \otimes w_3$. The *rank* $R(w)$ of $w \in W$ is defined as the minimum $r \in \mathbb{N}$ such that w can be written as a sum of r indecomposable tensors. We note that if $W_3 = \mathbb{C}$, then $R(w)$ is just the rank of the linear map $W_1^* \rightarrow W_2$ corresponding to w .

Strassen proved [32] that the minimum number of non-scalar multiplications sufficient to evaluate the bilinear map $W_1^* \times W_2^* \rightarrow W_3$ corresponding to w differs from $R(w)$ by at most a factor of two. Determining the rank of specific tensors turns out to be very difficult. Of particular interest are the tensors $(n, n, n) \in (\mathbb{C}^{n \times n})^* \otimes (\mathbb{C}^{n \times n})^* \otimes \mathbb{C}^{n \times n}$ describing the multiplication of two n by n matrices. The best known asymptotic upper bound [9] states $R(\langle n, n, n \rangle) = \mathcal{O}(n^{2.38})$, while the best known lower bound [2] is $R(\langle n, n, n \rangle) \geq \frac{5}{2}n^2 - 3n$.

The *border rank* $\underline{R}(w)$ of $w \in W$ is defined as the smallest $r \in \mathbb{N}$ such that w can be obtained as the limit of

a sequence $w_k \in W$ with $R(w_k) \leq r$ for all k . Clearly, $\underline{R}(w) \leq R(w)$. Border rank is a natural mathematical notion closely related to the rank and it has played an important role in the discovery of fast algorithms for matrix multiplication, see [5]. We note that the best known lower bound on the border rank of matrix multiplication [23] states that $\underline{R}(\langle n, n, n \rangle) \geq \frac{3}{2}n^2 + \frac{1}{2}n - 1$.

2.2 Orbit closure problem

It is possible to rephrase the determination of $\underline{R}(w)$ as an orbit closure problem. Consider the algebraic group

$$G := \mathrm{GL}(W_1) \times \mathrm{GL}(W_2) \times \mathrm{GL}(W_3) \quad (2.1)$$

acting linearly on the vector space $W = W_1 \otimes W_2 \otimes W_3$ via $(g_1, g_2, g_3)(w_1 \otimes w_2 \otimes w_3) := g_1(w_1) \otimes g_2(w_2) \otimes g_3(w_3)$. We shall denote by Gw the *orbit* of w and by \overline{Gw} its *orbit closure*. We say that w is a *degeneration* of v , written $w \trianglelefteq v$, iff $\overline{Gw} \subseteq \overline{Gv}$.

Suppose now $m \leq m_i$ and choose bases $e_1^i, \dots, e_{m_i}^i$ in each of the spaces W_i . The tensor

$$\langle m \rangle := \sum_{j=1}^m e_j^1 \otimes e_j^2 \otimes e_j^3, \quad (2.2)$$

is called an *m th unit tensor* in W . Another choice of bases leads to a tensor in the same G -orbit as $\langle m \rangle$, so that the orbit of m th unit tensors in W is a basis independent notion. It is easy to see that $\underline{R}(w) \leq m$ iff $w \trianglelefteq \langle m \rangle$, cf. [34].

3. THE GCT PROGRAM FOR TENSORS

We summarize here in a concise way the stepping stones of the GCT program [25, 26], adapted to the tensor setting. For this the review of the GCT program in [6] has been very helpful.

3.1 Semigroups of representations

For background on representation theory see [12, 14]. We denote by $V_{\lambda_i}(\mathrm{GL}(W_i))$ the Schur-Weyl module labelled by its highest weight $\lambda_i \in \mathbb{Z}^{m_i}$ (with monotonically decreasing entries). Those yield the rational irreducible G -modules

$$V_{\underline{\lambda}}(G) := V_{\lambda_1}(\mathrm{GL}(W_1)) \otimes V_{\lambda_2}(\mathrm{GL}(W_2)) \otimes V_{\lambda_3}(\mathrm{GL}(W_3)),$$

whose highest weights $\underline{\lambda}$ are triples $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$. We denote by $V_{\underline{\lambda}}(G)^* = V_{\underline{\lambda}^*}(G)$ the module dual to $V_{\underline{\lambda}}(G)$. Moreover, Λ_G^+ shall denote the semigroup of highest weights of G . For a dimension format \underline{m} we consider the subsemigroup $\Lambda^+(\underline{m}) := \bigcup_{d \in \mathbb{N}} \Lambda_d^+(\underline{m})$ of Λ_G^+ , where

$$\Lambda_d^+(\underline{m}) := \{ \underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \mid \lambda_i \vdash_{m_i} d, i = 1, 2, 3 \}.$$

Here we use the notation $\lambda_i \vdash_{m_i} d$ for a partition $\lambda_i = (\lambda_i^1, \dots, \lambda_i^{m_i})$ of d into at most m_i parts.

The action of G on W induces a linear action of G on the ring $\mathcal{O}(W)$ of polynomial functions on W via $(gf)(w) := f(g^{-1}w)$ for $g \in G$, $f \in \mathcal{O}(W)$, $w \in W$. For any tensor $w \in W$, this defines a linear action of G on the graded ring $\mathcal{O}(\overline{Gw}) = \bigoplus_{d \in \mathbb{N}} \mathcal{O}(\overline{Gw})_d$ of regular functions on \overline{Gw} . (By a regular function on \overline{Gw} we understand a restriction of a polynomial function.) Since G is reductive, the G -module $\mathcal{O}(\overline{Gw})_d$ splits into irreducible G -modules.

We define now the main objects of our investigations.

DEFINITION 3.1. *The semigroup of representations $S(w)$ of a tensor $w \in W$ is defined as*

$$S(w) := \{\underline{\lambda} \mid V_{\underline{\lambda}}(G)^* \text{ occurs in } \mathcal{O}(\overline{Gw})\}.$$

It is known that $S(w)$ is a finitely generated subsemigroup of $\Lambda^+(\underline{m})$, cf. [3]. It is easy to see that if $V_{\underline{\lambda}}(G)^*$ occurs in degree d , i.e., as a submodule of $\mathcal{O}(\overline{Gw})_d$, then $\underline{\lambda} \in \Lambda_d^+(\underline{m})$.

The general strategy of geometric complexity theory [25] is easily described. Schur's lemma implies that for $w, v \in W$

$$\overline{Gw} \subseteq \overline{Gv} \implies S(w) \subseteq S(v). \quad (3.1)$$

In particular, exhibiting some $\underline{\lambda} \in S(w) \setminus S(v)$ proves that \overline{Gw} is not contained in \overline{Gv} . If $v = \langle m \rangle$, this establishes the lower bound $\underline{R}(w) > m$. We call such $\underline{\lambda}$ a *representation theoretic obstruction*. We note that a more refined approach would be to study the multiplicity of $V_{\underline{\lambda}}(G)^*$ in $\mathcal{O}(\overline{Gw})$, which can only decrease under degenerations.

3.2 Kronecker semigroup

Let $[\lambda_i]$ denote the irreducible representation of the symmetric group S_d on d letters labelled by the partition $\lambda_i \vdash d$. For $\underline{\lambda} \in \Lambda_d^+(\underline{m})$ we define the *Kronecker coefficient* $g(\underline{\lambda})$ as the dimension of the space of S_d -invariants of the tensor product $[\lambda_1] \otimes [\lambda_2] \otimes [\lambda_3]$. It is a well known fact that

$$g(\underline{\lambda}) = \text{mult}(V_{\underline{\lambda}}(G)^*, \mathcal{O}(W)_d), \quad (3.2)$$

see [22]. The *Kronecker semigroup* of format \underline{m} is defined by $K(\underline{m}) := \bigcup_{d \in \mathbb{N}} \{\underline{\lambda} \in \Lambda_d^+(\underline{m}) \mid g(\underline{\lambda}) \neq 0\}$.

LEMMA 3.2. *We have $S(w) \subseteq K(\underline{m})$ with equality holding for Zariski almost all $w \in W$.*

3.3 Inheritance

For applying the criterion " $\underline{R}(w) \leq m$ iff $w \trianglelefteq \langle m \rangle$ " from §2.2, we need to understand how $S(w)$ changes when we embed $w \in W$ in a larger space. Fortunately, when properly interpreted, nothing happens. Suppose that W_i is a subspace of W'_i , put $m'_i := \dim W'_i$, and let

$$W' := W'_1 \otimes W'_2 \otimes W'_3, \quad G' := \text{GL}(W'_1) \times \text{GL}(W'_2) \times \text{GL}(W'_3).$$

Let w' denote the image of $w \in W$ under the embedding $W \hookrightarrow W'$. A highest G -weight $\underline{\lambda}$ with nonnegative entries can be interpreted as a highest G' -weight $\underline{\lambda}$ by appending zeros to the partitions λ_i . We may thus interpret $\Lambda^+(\underline{m})$ as a subset of $\Lambda^+(\underline{m}')$.

PROPOSITION 3.3. *With the above conventions we have $S(w) = S(w')$.*

This result can be shown similarly as in [26, 22, 6].

3.4 Stabilizer and invariants

As a first approach towards understanding $S(w)$ we may replace the orbit closure \overline{Gw} by the orbit Gw and focus on the representations occurring in the ring $\mathcal{O}(Gw)$ of regular functions on Gw . (A regular function on Gw is a function that is locally rational, cf. [15, p. 15].) This leads to definition of the *auxiliary semigroup of representations*:

$$S^o(w) := \{\underline{\lambda} \in \Lambda_G^+ \mid V_{\underline{\lambda}}(G)^* \text{ occurs in } \mathcal{O}(Gw)\}.$$

$S^o(w)$ is finitely generated [3] and clearly contains $S(w)$.

The *stabilizer* of w is defined as $H := \text{stab}(w) := \{g \in G \mid gw = w\}$. Let $V_{\underline{\lambda}}(G)^H$ denote the space of H -invariants in $V_{\underline{\lambda}}(G)$. The next characterization follows from the algebraic Peter-Weyl Theorem for G as in [6].

PROPOSITION 3.4. $S^o(w) = \{\underline{\lambda} \in \Lambda_G^+ \mid V_{\underline{\lambda}}(G)^H \neq 0\}$.

3.5 Stabilizers of associative algebras

Let $\text{Bil}(U, V; W)$ denote the space of bilinear maps $U \times V \rightarrow W$, where U, V, W are finite dimensional vector spaces. The group $G = \text{GL}(U) \times \text{GL}(V) \times \text{GL}(W)$ acts on $\text{Bil}(U, V; W)$ via $(\alpha, \beta, \gamma) \cdot \varphi := \gamma \varphi(\alpha^{-1} \times \beta^{-1})$. By definition, $\text{GL}(U)$ acts on the dual module U^* via $\alpha \cdot \ell := (\alpha^{-1})^*(\ell)$ for $\alpha \in \text{GL}(U)$, $\ell \in U^*$. It is straightforward to check that the canonical isomorphism $U^* \otimes V^* \otimes W \rightarrow \text{Bil}(U, V; W)$ is G -equivariant.

LEMMA 3.5. *Let $\varphi \in \text{Bil}(U, V; W)$ and $w \in U^* \otimes V^* \otimes W$ be the corresponding tensor. Then*

$$\text{stab}(w) = \{((\alpha^{-1})^*, (\beta^{-1})^*, \gamma) \mid \forall u, v \varphi(\alpha(u), \beta(v)) = \gamma(\varphi(u, v))\}$$

Now let A be a finite dimensional associative \mathbb{C} -algebra with 1. Its multiplication map $A \times A \rightarrow A$ corresponds to a tensor $w_A \in A^* \otimes A^* \otimes A$. We denote by A^\times the unit group of A and by $\text{Aut}A$ its group of algebra automorphisms. For $a \in A$ we denote by $L_a: A \rightarrow A, x \mapsto ax$ the left multiplication with a . Similarly, R_a denotes the right multiplication with a .

The following observation goes back to [10].

LEMMA 3.6. *We have*

$$\text{stab}(w_A) = \left\{ (L_{\varepsilon^{-1}}^*(\psi^{-1})^*, R_{\eta^{-1}}^*(\psi^{-1})^*, L_{\varepsilon} R_{\eta} \psi) \mid \varepsilon, \eta \in A^\times, \psi \in \text{Aut}A \right\}.$$

PROOF. Let $\alpha, \beta, \gamma \in \text{GL}(A)$. Suppose that $((\alpha^{-1})^*, (\beta^{-1})^*, \gamma) \in \text{stab}(w_A)$. By Lemma 3.5 we have $\alpha(a)\beta(b) = \gamma(ab)$ for all $a, b \in A$. Setting $a = 1$ and $b = 1$, respectively, we get $\alpha(a)\beta(1) = \gamma(a)$ and $\alpha(1)\beta(b) = \gamma(b)$. Hence $\varepsilon := \alpha(1)$ and $\eta := \beta(1)$ must be units of A . We define now $\psi(a) := \varepsilon^{-1}\gamma(a)\eta^{-1}$. Then we have $\psi(1) = 1$ and

$$\begin{aligned} \psi(a)\psi(b) &= \varepsilon^{-1}\gamma(a)\eta^{-1} \varepsilon^{-1}\gamma(b)\eta^{-1} = \varepsilon^{-1}\alpha(a)\beta(b)\eta^{-1} \\ &= \varepsilon^{-1}\gamma(ab)\eta^{-1} = \psi(ab). \end{aligned}$$

Therefore $\psi \in \text{Aut}A$. By construction, $\alpha = L_{\varepsilon}\psi$, $\beta = R_{\eta}\psi$, and $\gamma = L_{\varepsilon}R_{\eta}\psi$, and hence $(\alpha^{-1})^* = L_{\varepsilon^{-1}}^*(\psi^{-1})^*$, $(\beta^{-1})^* = R_{\eta^{-1}}^*(\psi^{-1})^*$. The argument is reversible. \square

3.6 Stability

Consider the subgroup $G_s := \text{SL}(W_1) \times \text{SL}(W_2) \times \text{SL}(W_3)$ of G .

DEFINITION 3.7. *A tensor $w \in W$ is called stable, iff $G_s w$ is closed.*

Consider the residue class map

$$\prod_{i=1}^3 \mathbb{Z}^{m_i} \rightarrow \prod_{i=1}^3 \mathbb{Z}^{m_i} / \mathbb{Z}\varepsilon_{m_i},$$

where $\varepsilon_{m_i} := (1, \dots, 1)$. When interpreting highest weights of G_s -modules appropriately, this defines a surjective morphism $\pi: \Lambda_{G_s}^+ \rightarrow \Lambda_{G_s}^+$ of the semigroup of highest weights of G and G_s , respectively.

We put $S_s(w) := \pi(S(w))$ and $S_s^o(w) := \pi(S^o(w))$. These semigroups describe the irreducible G_s -modules occurring in $\mathcal{O}(\overline{Gw})$ and $\mathcal{O}(Gw)$, respectively. However, when going over to G_s -modules, the information about the degree d in which the modules occur is lost.

PROPOSITION 3.8. *If w is stable, then $S_s(w) = S_s^o(w)$.*

PROOF. Put $\varepsilon_{\underline{m}} := (\varepsilon_{m_1}, \varepsilon_{m_2}, \varepsilon_{m_3})$. The assertion is equivalent to the statement

$$\forall \underline{\lambda} \in S^o(w) \exists k \in \mathbb{Z} \quad \underline{\lambda} + k\varepsilon_{\underline{m}} \in S(w). \quad (3.3)$$

Suppose that $\underline{\lambda} \in S^o(w)$. Then $V_{\underline{\lambda}}(G)^*$ occurs in $\mathcal{O}(Gw)_d$ for some $d \in \mathbb{Z}$. Let $f \in \mathcal{O}(Gw)_d$ be a highest weight vector of $V_{\underline{\lambda}}(G)^*$. The restriction \tilde{f} of f to $G_s w$ does not vanish since Gw is the cone generated by $G_s w$. So \tilde{f} is a highest weight vector and $V_{\pi(\underline{\lambda})}(G_s)^*$ occurs in $\mathcal{O}(G_s w)$.

The G_s -equivariant restriction morphism $\mathcal{O}(\overline{Gw}) \rightarrow \mathcal{O}(G_s w)$ is surjective since $G_s w$ is assumed to be closed. It follows that $\mathcal{O}(\overline{Gw})$ contains an irreducible module $V_{\pi(\underline{\lambda})}(G_s)^*$. This means that $V_{\underline{\lambda} + k\varepsilon_{\underline{m}}}(G)^*$ occurs in $\mathcal{O}(\overline{Gw})$ for some $k \in \mathbb{Z}$. \square

Combining this with Proposition 3.4, we obtain a characterization of $S_s(w)$ for stable tensors w , which only involves the stabilizer H of w . The problem is reduced to the question of which $V_{\underline{\lambda}}(G)$ contain nonzero H -invariants.

We need some criterion for testing stability. By a *one-parameter subgroup* of G_s we understand a morphism $\sigma: \mathbb{C}^\times \rightarrow G_s$ of algebraic groups. The *centralizer* $Z_{G_s}(R_s)$ of a subgroup R_s of G_s is defined as the set of $g \in G_s$ such that $gh = hg$ for all $h \in R_s$. For instance, let T_s denote the maximal torus of G_s . Then we have $Z_{G_s}(T_s) = T_s$.

The following important stability criterion is a consequence Kempf's [17] refinement of the Hilbert-Mumford criterion.

THEOREM 3.9. *Let $w \in W$ be a tensor and R_s be a reductive subgroup of G_s contained in the stabilizer of w . We assume that for all one-parameter subgroups σ of G_s , with image in the centralizer $Z_{G_s}(R_s)$, the limit $\lim_{t \rightarrow 0} \sigma(t)w$ lies in the G_s -orbit of w , provided the limit exists. Then w is stable.*

4. UNIT TENSORS

4.1 Stabilizer and stability

Suppose $W_i = \mathbb{C}^m$, $G_m := \mathrm{GL}_m \times \mathrm{GL}_m \times \mathrm{GL}_m$, and recall the definition of the m th unit tensor $\langle m \rangle$ from (2.2). Let $P_\pi \in \mathrm{GL}_m$ denote the permutation matrix corresponding to $\pi \in S_m$.

PROPOSITION 4.1. *The stabilizer H_m of $\langle m \rangle$ is the semidirect product of the normal divisor*

$$D_m := \{(\mathrm{diag}(a), \mathrm{diag}(b), \mathrm{diag}(c)) \mid \forall i \ a_i b_i c_i = 1\},$$

and the symmetric group S_m diagonally embedded in G_m via $\pi \mapsto (P_\pi, P_\pi, P_\pi)$.

PROOF. Let S_m denote the diagonal embedding of the symmetric group in G_m . Obviously, $D_m \cap S_m = \{\mathrm{id}\}$. It is easy to see that S_m normalizes D_m . Hence $D_m S_m$ is a subgroup of G_m and D_m is a normal divisor of $D_m S_m$. It remains to prove that the stabilizer H_m equals $D_m S_m$. The inequality $D_m S_m \subseteq H_m$ is obvious.

Note that $\langle m \rangle$ is the structural tensor of the algebra $A = \mathbb{C}^m$. It is straightforward to check that $\mathrm{Aut} A = \{P_\pi \mid \pi \in S_m\}$. Note that $(P_\pi^{-1})^* = P_\pi$. Hence Lemma 3.6 implies

$$H_m = \mathrm{stab}(\langle m \rangle) = \{(\mathrm{diag}(\varepsilon^{-1})P_\pi, \mathrm{diag}(\eta^{-1})P_\pi, \mathrm{diag}(\varepsilon\eta)P_\pi) \mid \varepsilon, \eta \in (\mathbb{C}^\times)^m, \pi \in S_m\}$$

and we obtain $H_m = D_m S_m$. \square

We remark that $\langle m \rangle$ is uniquely determined by its stabilizer up to a scalar. Using Theorem 3.9 one easily proves the following.

PROPOSITION 4.2. *The unit tensor $\langle m \rangle$ is stable.*

4.2 Representations

Let $\mathrm{Par}_m(d)$ denote the set of partitions of d into at most m parts. The *dominance order* \preceq on $\mathrm{Par}_m(d)$ is defined by $\lambda \preceq \mu$ iff $\sum_{j=1}^k \lambda^j \leq \sum_{j=1}^k \mu^j$ for all k . This defines a lattice, in particular two partitions λ, μ have a well defined meet $\lambda \wedge \mu$, cf. [31]. We call $\alpha \in \mathrm{Par}_m(d)$ *regular* if its components are pairwise distinct.

LEMMA 4.3. (1) *The set of regular partitions in $\mathrm{Par}_m(d)$ has a unique smallest element $\perp_m(d)$. (2) For any $\lambda \in \mathrm{Par}_m(d)$ we have $\perp_{m+1}(d + km) \preceq (\lambda^1 + k, \dots, \lambda^m + k, 0)$ for sufficiently large k .*

Let T_m denote the maximal torus of GL_m of diagonal matrices. For $\alpha \in \mathbb{Z}^m$ with $|\alpha| := \sum_j \alpha_j = d$ and $\lambda \in \mathrm{Par}_m(d)$ one defines the *weight space* of $V_\lambda = V_\lambda(\mathrm{GL}_m)$ for the weight α as

$$V_\lambda^\alpha := \{v \in V_\lambda \mid \forall t \in T_m \ t \cdot v = t^\alpha v\}.$$

Here we used the shorthand notation $t = \mathrm{diag}(t_1, \dots, t_m)$ and $t^\alpha = t_1^{\alpha_1} \dots t_m^{\alpha_m}$. It is well known that V_λ decomposes as $V_\lambda = \bigoplus_\alpha V_\lambda^\alpha$. Moreover, for $\alpha \in \mathrm{Par}_m(d)$, V_λ^α is nonzero iff $\alpha \preceq \lambda$, cf. [12].

The symmetric group S_m acts on \mathbb{Z}^m by permutation, namely $(\pi\alpha)(i) := \alpha(\pi^{-1}(i))$ for $\pi \in S_m$. It is easy to check that $P_\pi V_\lambda^\alpha = V_\lambda^{\pi\alpha}$. In particular, the stabilizer $\mathrm{stab}(\alpha)$ of α leaves V_λ^α invariant. Note that $\mathrm{stab}(\alpha)$ is trivial iff α is regular.

THEOREM 4.4. *For $\underline{\lambda} \in \Lambda_d(m, m, m)$ we have $\dim(V_{\underline{\lambda}})^{H_m} = \sum_\alpha \dim(V_{\lambda_1}^\alpha \otimes V_{\lambda_2}^\alpha \otimes V_{\lambda_3}^\alpha)^{\mathrm{stab}(\alpha)}$, where the sum is over all $\alpha \in \mathrm{Par}_m(d)$ such that $\alpha \preceq \lambda_1 \wedge \lambda_2 \wedge \lambda_3$ and H_m denotes the stabilizer of $\langle m \rangle$.*

PROOF. Let $\underline{\lambda} \in \Lambda_d(m, m, m)$. The weight decomposition

$$V_{\underline{\lambda}} = V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} = \bigoplus_{\alpha, \beta, \gamma} V_{\lambda_1}^\alpha \otimes V_{\lambda_2}^\beta \otimes V_{\lambda_3}^\gamma$$

yields $(V_{\underline{\lambda}})^{D_m} = \bigoplus_{\alpha, \beta, \gamma} (V_{\lambda_1}^\alpha \otimes V_{\lambda_2}^\beta \otimes V_{\lambda_3}^\gamma)^{D_m}$, cf. Proposition 4.1. We claim that

$$(V_{\lambda_1}^\alpha \otimes V_{\lambda_2}^\beta \otimes V_{\lambda_3}^\gamma)^{D_m} = \begin{cases} V_{\lambda_1}^\alpha \otimes V_{\lambda_2}^\alpha \otimes V_{\lambda_3}^\alpha & \text{if } \alpha = \beta = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, let $v \in (V_{\lambda_1}^\alpha \otimes V_{\lambda_2}^\beta \otimes V_{\lambda_3}^\gamma)^{D_m}$ be nonzero. For $t = (\mathrm{diag}(a), \mathrm{diag}(b), \mathrm{diag}(c)) \in D_m$ we obtain $v = tv = a^\alpha b^\beta c^\gamma v = a^{\alpha-\gamma} b^{\beta-\gamma} v$, using $a_i b_i c_i = 1$. Since the $a_i, b_i \in \mathbb{C}^\times$ are arbitrary, we infer $\alpha = \beta = \gamma$. The argument can be reversed.

We put now $A := \{\alpha \in \mathbb{Z}^m \mid |\alpha| = d, \alpha \preceq \lambda_1 \wedge \lambda_2 \wedge \lambda_3\}$, $M^\alpha := V_{\lambda_1}^\alpha \otimes V_{\lambda_2}^\alpha \otimes V_{\lambda_3}^\alpha$ and note that $M^\alpha \neq 0$ for all $\alpha \in A$. We have just seen that $(V_{\underline{\lambda}})^{D_m} = \bigoplus_{\alpha \in A} M^\alpha$.

The set A is invariant under the S_m -action and its orbits intersect $\mathrm{Par}_m(d)$ in exactly one partition. We note that $\pi M^\alpha = M^{\pi\alpha}$ for $\pi \in S_m$. Let \mathcal{B} denote the set of orbits and put $M_B := \bigoplus_{\alpha \in B} M^\alpha$ for $B \in \mathcal{B}$. Then $(V_{\underline{\lambda}})^{D_m} = \bigoplus_{B \in \mathcal{B}} M_B$. Proposition 4.1 tells us $H_m = D_m S_m$ and hence

$$(V_{\underline{\lambda}})^{H_m} = ((V_{\underline{\lambda}})^{D_m})^{S_m} = \bigoplus_{B \in \mathcal{B}} (M_B)^{S_m}$$

using that the M_B are S_m -invariant. In order to complete the proof it suffices to show that

$$\dim(M_B)^{S_m} = \dim(M^\alpha)^{\text{stab}(\alpha)}$$

for $B = S_m \alpha$, $\alpha \in A \cap \text{Par}_m(d)$. For proving this, we fix $\alpha \in A \cap \text{Par}_m(d)$ and write $H := \text{stab}(\alpha)$. Let π_1, \dots, π_t be a system of representatives for the left cosets of H in S_m with $\pi_1 = \text{id}$. So $S_m = \pi_1 H \cup \dots \cup \pi_t H$. Then the S_m -orbit of α equals $S_m \alpha = \{\pi_1 \alpha, \dots, \pi_t \alpha\}$. Consider

$$M_B = \bigoplus_{j=1}^t \pi_j M^\alpha$$

and the corresponding projection $p: M_B \rightarrow M^\alpha$. Suppose that $v = \sum_j v_j \in (M_B)^{S_m}$ with $v_j \in \pi_j M^\alpha$. Since the spaces $\pi_1 M^\alpha, \dots, \pi_t M^\alpha$ are permuted by the action of S_m , we derive from $v = \pi_k v = \sum_j \pi_k v_j$ that $v_j = \pi_j v_1$. Moreover, since $\sigma \in H$ fixes M^α and permutes the spaces $\pi_2 M^\alpha, \dots, \pi_t M^\alpha$, we obtain $\sigma v_1 = v_1$. Therefore, $(M_B)^{S_m} \rightarrow (M^\alpha)^H, v \mapsto p(v) = v_1$ is well defined and injective. We claim that this map is also surjective.

For showing this, let $v_1 \in (M^\alpha)^H$, set $v_j := \pi_j v_1$, and put $v := \sum_j v_j$. Clearly, $p(v) = v_1$. Fix $\sigma \in H$ and i . For any j there is a unique $k = k(j)$ such that $\sigma \pi_i \pi_j H = \pi_k H$. Moreover, $j \mapsto k(j)$ is a permutation of $\{1, \dots, t\}$. Using the H -invariance of v_1 we obtain that $\sigma \pi_i v_j = \sigma \pi_i \pi_j v_1 = \pi_k v_1 = v_k$. Therefore $\sigma \pi_i v = \sum_k v_k = v$. Thus $v \in (M_B)^{S_m}$. \square

The next result shows that the highest weights outside the auxiliary semigroup of the unit tensors are very rare.

COROLLARY 4.5. (1) *If there is a regular $\alpha \preceq \lambda_1 \wedge \lambda_2 \wedge \lambda_3$, then $\underline{\lambda} \in S^o(\langle m \rangle)$.*

(2) *If $\lambda_1, \lambda_2, \lambda_3$ are all regular, then $\underline{\lambda} \in S^o(\langle m \rangle)$.*

PROOF. (1) is an immediate consequence of Proposition 3.4 and Theorem 4.4. If $\lambda_1, \lambda_2, \lambda_3$ are all regular, then $\perp_m(d) \preceq \lambda_i$ for $i = 1, 2, 3$ by Lemma 4.3(1). Now apply (1). \square

THEOREM 4.6. *For any $\underline{\lambda} \in \Lambda_d^+(m, m, m)$ there exists $k \in \mathbb{N}$ such that $\underline{\lambda}' \in S^o(\langle m+1 \rangle)$, where $\lambda_i = (\lambda_i^1, \dots, \lambda_i^m)$ and $\lambda_i' = (\lambda_i^1 + k, \dots, \lambda_i^m + k, 0)$.*

PROOF. Lemma 4.3(2) implies $\perp_{m+1}(d+km) \preceq \lambda_1', \lambda_2', \lambda_3'$ for sufficiently large k . Now apply Corollary 4.5(1). \square

Theorem 4.6 has severe consequences. It tells us that for any tensor w of format (m, m, m) , the trivial lower bound $\underline{R}(w) > m$ is the best that can be shown using G_s -obstructions!

5. MATRIX MULTIPLICATION TENSORS

We fix complex vector spaces U_i of dimension n_i , put $W_{12} := U_1^* \otimes U_2$, $W_{23} := U_2^* \otimes U_3$, $W_{31} := U_3^* \otimes U_1$, and consider the group

$$G := \text{GL}(U_1^* \otimes U_2) \times \text{GL}(U_2^* \otimes U_3) \times \text{GL}(U_3^* \otimes U_1).$$

acting on $W := W_{12} \otimes W_{23} \otimes W_{31}$. We define the *matrix multiplication tensor* $M_{\underline{U}} \in W := W_{12} \otimes W_{23} \otimes W_{31}$ as the tensor corresponding to the linear form

$$W^* \rightarrow \mathbb{C}, u_1 \otimes \ell_2 \otimes u_2 \otimes \ell_3 \otimes u_3 \otimes \ell_1 \mapsto \ell_1(u_1) \ell_2(u_2) \ell_3(u_3),$$

obtained as the product of three contractions ($\ell_i \in U_i^*$ and $u_i \in U_i$). To justify the naming we note that, using the canonical isomorphisms $\text{Hom}(U_2, U_1) \simeq U_1 \otimes U_2^*$ and

$\text{Bil}(U, V; W) \simeq U^* \otimes V^* \otimes W$, one easily checks that $M_{\underline{U}}$ corresponds to the bilinear map

$$\begin{aligned} M_{\underline{U}}: \text{Hom}(U_2, U_1) \times \text{Hom}(U_3, U_2) &\rightarrow \text{Hom}(U_3, U_1), \\ (\varphi, \psi) &\mapsto \varphi \circ \psi \end{aligned}$$

describing the composition of linear maps (note that we exchanged the order for the third factor: $\text{Hom}(U_3, U_1) \simeq U_3^* \otimes U_1$). If $U_i = \mathbb{C}^{n_i}$, then this bilinear map corresponds to the multiplication of $n_1 \times n_2$ with $n_2 \times n_3$ matrices. In this case we shall write $\langle n_1, n_2, n_3 \rangle = M_{\underline{U}}$.

5.1 Stabilizer and stability

We put $K := \text{GL}(U_1) \times \text{GL}(U_2) \times \text{GL}(U_3)$ and consider the following morphism of groups

$$\Phi: K \rightarrow G, \tag{5.1}$$

$$(a_1, a_2, a_3) \mapsto ((a_1^*)^{-1} \otimes a_2, (a_2^*)^{-1} \otimes a_3, (a_3^*)^{-1} \otimes a_1)$$

with the kernel $\mathbb{C}^\times \cdot \text{id} \simeq \mathbb{C}^\times$. Note that $\text{GL}(U_i)$ acts on $U_i^* \otimes U_i$ in the following way:

$$\begin{aligned} a_i \cdot (\ell_i \otimes u_i) &= ((a_i^{-1})^* \otimes a_i)(\ell_i \otimes u_i) = (a_i^{-1})^*(\ell_i) \otimes a_i(u_i) \\ &= (\ell_i \circ a_i^{-1}) \otimes a_i(u_i). \end{aligned}$$

Hence this action leaves the trace $U_i^* \otimes U_i \rightarrow \mathbb{C}, \ell_i \otimes u_i \mapsto \ell_i(u_i)$ invariant. This implies that the image of Φ is contained in the stabilizer H of $M_{\underline{U}}$. In fact, equality holds.

PROPOSITION 5.1. *The stabilizer H of $M_{\underline{U}}$ equals the image of Φ . In particular, $H \simeq K/\mathbb{C}^\times$.*

PROOF. We provide the proof in the cubic case only and thus assume $U_i = \mathbb{C}^n$. The matrix multiplication tensor $M_{\underline{U}}$ is the structural tensor of the associative algebra $A = \text{End}(U)$. Note that $A^\times = \text{GL}(U)$. Recall that $L_a, R_b: A \rightarrow A$ denote the left multiplication with a and the right multiplication with b , respectively ($a, b \in A$). If we interpret $A = U \otimes U^*$, then we have $L_a R_b = a \otimes b^*$. Lemma 3.6 states that any element g of $\text{stab}(M_{\underline{U}})$ is of the form

$$g = (L_{\varepsilon^{-1}}(\psi^{-1})^*, R_{\eta^{-1}}(\psi^{-1})^*, L_\varepsilon R_\eta \psi)$$

for some $\varepsilon, \eta \in A^\times, \psi \in \text{Aut} A$. The Skolem-Noether Theorem [16] implies that any automorphism ψ of A is of the form $\psi = L_\rho R_\rho^{-1}$ for some $\rho \in A^\times$. We thus obtain

$$\psi^{-1} L_{\varepsilon^{-1}} = L_{\rho^{-1}} R_\rho L_{\varepsilon^{-1}} = L_{\rho^{-1} \varepsilon^{-1}} R_\rho = \rho^{-1} \varepsilon^{-1} \otimes \rho^*,$$

which implies $L_{\varepsilon^{-1}}(\psi^{-1})^* = (\psi^{-1} L_{\varepsilon^{-1}})^* = ((\varepsilon \rho)^{-1})^* \otimes \rho$. Similarly, we obtain $R_{\eta^{-1}}(\psi^{-1})^* = (\rho^{-1})^* \otimes \eta^{-1} \rho$. Finally, $L_\varepsilon R_\eta \psi = \varepsilon \rho \otimes (\rho^{-1} \eta)^* \simeq (\rho^{-1} \eta)^* \otimes \varepsilon \rho$. (The flip \simeq is due to our convention $\text{Hom}(U_3, U_1) \simeq U_3^* \otimes U_1$, unlike $\text{Hom}(U_2, U_1) \simeq U_1 \otimes U_2^*$, $\text{Hom}(U_3, U_2) \simeq U_2 \otimes U_3^*$.) Setting $\alpha_1 = \varepsilon \rho, \alpha_2 = \rho, \alpha_3 = \eta^{-1} \rho$ we see that g has the required form. \square

We remark that $M_{\underline{U}}$ is uniquely determined by its stabilizer up to a scalar.

PROPOSITION 5.2. *The matrix multiplication tensor $M_{\underline{U}}$ is stable.*

PROOF. We follow [24, Proposition 5.2.1]. Assume that $U_i = \mathbb{C}^{n_i}$. Let $T(K_s)$ and T_s denote the maximal tori of $K_s := \text{SL}(U_1) \times \text{SL}(U_2) \times \text{SL}(U_3)$ and G_s , respectively, consisting of triples of diagonal matrices with determinant 1. It

is clear that $R_s := \Phi(T(K_s))$ is a subgroup of T_s . Since R_s is a connected subgroup of a torus, it is itself a torus and thus reductive [19].

We claim that T_s equals the centralizer of R_s in G_s . Indeed suppose that $g = (g_1, g_2, g_3) \in G_s$ commutes with all elements of R_s . Then g_1 commutes with all diagonal matrices $\text{diag}(a_i b_j^{-1})$, where $a_1 \cdots a_{n_1} = 1$ and $b_1 \cdots b_{n_2} = 1$. It is possible to choose a_i, b_j such that $a_i b_j^{-1}$ are pairwise distinct. Therefore g_1 must be a diagonal matrix. Similarly, g_2, g_3 must be diagonal so that $g \in G_s$.

We apply now Theorem 3.9 to the reductive subgroup R_s of the stabilizer H of $M_{\underline{U}}$. Any one-parameter subgroup $\sigma: \mathbb{C}^\times \rightarrow T_s$ is of the form $\sigma(t) = (\sigma_1(t), \sigma_2(t), \sigma_3(t))$ with

$$\sigma_1(t) = \text{diag}(t^{\mu_{ij}}), \quad \sigma_2(t) = \text{diag}(t^{\nu_{jk}}), \quad \sigma_3(t) = \text{diag}(t^{\pi_{ki}}),$$

where $\mu_{ij}, \nu_{jk}, \pi_{ki} \in \mathbb{Z}$ for $i \leq n_1, j \leq n_2, k \leq n_3$. Since $\det \sigma_1(t) = \det \sigma_2(t) = \det \sigma_3(t) = 1$ we must have

$$\sum_{i,j} \mu_{ij} = 0, \quad \sum_{j,k} \nu_{jk} = 0, \quad \sum_{k,i} \pi_{ki} = 0. \quad (5.2)$$

Let $(e_{ij}), (e_{jk}), (e_{ki})$ denote the standard bases of $\mathbb{C}^{n_1 \times n_2}, \mathbb{C}^{n_2 \times n_3}, \mathbb{C}^{n_3 \times n_1}$, respectively. The matrix multiplication tensor can then be expressed as

$$\langle n_1, n_2, n_3 \rangle = \sum_{i,j,k} e_{ij} \otimes e_{jk} \otimes e_{ki}.$$

We have

$$\sigma(t) \langle n_1, n_2, n_3 \rangle = \sum_{i,j,k} t^{\mu_{ij} + \nu_{jk} + \pi_{ki}} e_{ij} \otimes e_{jk} \otimes e_{ki}.$$

Suppose that the limit of $\sigma(t) \langle n_1, n_2, n_3 \rangle$ for $t \rightarrow 0$ exist. Then

$$\forall i, j, k \quad \mu_{ij} + \nu_{jk} + \pi_{ki} \geq 0.$$

Summing over all i, j, k and using (5.2) we get

$$\sum_{i,j,k} (\mu_{ij} + \nu_{jk} + \pi_{ki}) = \sum_k \sum_{ij} \mu_{ij} + \sum_i \sum_{jk} \nu_{jk} + \sum_j \sum_{ki} \pi_{ki} = 0.$$

Therefore, we have $\mu_{ij} + \nu_{jk} + \pi_{ki} = 0$ for all i, j, k . We conclude that $\lim_{t \rightarrow 0} \sigma(t) \langle n_1, n_2, n_3 \rangle = \langle n_1, n_2, n_3 \rangle$. Theorem 3.9 implies that the G_s -orbit of $\langle n_1, n_2, n_3 \rangle$ is closed. \square

5.2 Representations

Suppose that $\lambda_{12} \in \mathbb{Z}^{n_1 n_2}$ is a highest weight vector for $\text{GL}(U_1^* \otimes U_2)$ and $\lambda_{23} \in \mathbb{Z}^{n_2 n_3}, \lambda_{31} \in \mathbb{Z}^{n_3 n_1}$ are highest weight vectors for $\text{GL}(U_2^* \otimes U_3)$ and $\text{GL}(U_3^* \otimes U_1)$, respectively. Put $\underline{\lambda} = (\lambda_{12}, \lambda_{23}, \lambda_{31})$ and consider the irreducible G -module $V_{\underline{\lambda}} :=$

$$V_{\lambda_{12}}(\text{GL}(U_1^* \otimes U_2)) \otimes V_{\lambda_{23}}(\text{GL}(U_2^* \otimes U_3)) \otimes V_{\lambda_{31}}(\text{GL}(U_3^* \otimes U_1)).$$

THEOREM 5.3. *Let $\lambda_{12}, \lambda_{23}, \lambda_{31}$ be partitions of d and H be the stabilizer of $M_{\underline{U}}$. Then*

$$\dim(V_{\underline{\lambda}})^H = \sum_{\substack{\mu_1 \vdash_{n_1} d \\ \mu_2 \vdash_{n_2} d \\ \mu_3 \vdash_{n_3} d}} g(\lambda_{12}, \mu_1, \mu_2) \cdot g(\lambda_{23}, \mu_2, \mu_3) \cdot g(\lambda_{31}, \mu_3, \mu_1).$$

PROOF. The group morphisms

$$\begin{aligned} \Gamma_{12}: \text{GL}(U_1^*) \times \text{GL}(U_2) &\rightarrow \text{GL}(U_1^* \otimes U_2), & (a^*, b) &\mapsto a^* \otimes b \\ \Gamma_{23}: \text{GL}(U_2^*) \times \text{GL}(U_3) &\rightarrow \text{GL}(U_2^* \otimes U_3), & (b^*, c) &\mapsto b^* \otimes c \\ \Gamma_{31}: \text{GL}(U_3^*) \times \text{GL}(U_1) &\rightarrow \text{GL}(U_3^* \otimes U_1), & (c^*, a) &\mapsto c^* \otimes a \end{aligned}$$

combine to a morphism $\Gamma: \Pi \rightarrow G$, where Π denotes the group

$$\Pi := \text{GL}(U_1^*) \times \text{GL}(U_2) \times \text{GL}(U_2^*) \times \text{GL}(U_3) \times \text{GL}(U_3^*) \times \text{GL}(U_1).$$

Moreover, we have the group morphisms

$$\Lambda_i: \text{GL}(U_i) \rightarrow \text{GL}(U_i^*) \times \text{GL}(U_i), \quad a_i \mapsto ((a_i^*)^{-1}, a_i)$$

combining to a morphism (note the permutation)

$$\Lambda: K \rightarrow \Pi, \quad (a_1, a_2, a_3) \mapsto ((a_1^*)^{-1}, a_2, (a_2^*)^{-1}, a_3, (a_3^*)^{-1}, a_1).$$

We have thus factored the morphism $\Phi: K \rightarrow G$ as $\Phi = \Gamma \circ \Lambda$, cf. (5.1). Proposition 5.1 states that $H = \text{im} \Phi$. In order to determine $\dim(V_{\underline{\lambda}})^H$, we first describe the splitting of $V_{\underline{\lambda}}$ into irreducible Π -modules with respect to Γ and then, in a second step, extract their K -invariants.

For the first step, note that, upon restriction with respect to Γ_{12} , we have the decomposition $V_{\lambda_{12}}(\text{GL}(U_1^* \otimes U_2)) =$

$$\bigoplus_{\mu_1, \tilde{\mu}_2} g(\lambda_{12}, \mu_1, \tilde{\mu}_2) V_{\mu_1}(\text{GL}(U_1^*)) \otimes V_{\tilde{\mu}_2}(\text{GL}(U_2)),$$

where the sum is over all partitions $\mu_1 \vdash_{n_1} d, \tilde{\mu}_2 \vdash_{n_2} d$. For this characterization of the Kronecker coefficients g see [31, (7.221), p. 537]. Similarly,

$$V_{\lambda_{23}}(\text{GL}(U_2^* \otimes U_3)) =$$

$$\bigoplus_{\mu_2, \tilde{\mu}_3} g(\lambda_{23}, \mu_2, \tilde{\mu}_3) V_{\mu_2}(\text{GL}(U_2^*)) \otimes V_{\tilde{\mu}_3}(\text{GL}(U_3)),$$

$$V_{\lambda_{31}}(\text{GL}(U_3^* \otimes U_1)) =$$

$$\bigoplus_{\mu_3, \tilde{\mu}_1} g(\lambda_{31}, \mu_3, \tilde{\mu}_1) V_{\mu_3}(\text{GL}(U_3^*)) \otimes V_{\tilde{\mu}_1}(\text{GL}(U_1)),$$

where the sums are over all $\mu_2 \vdash_{n_2} d, \tilde{\mu}_3 \vdash_{n_3} d$ and $\mu_3 \vdash_{n_3} d, \tilde{\mu}_1 \vdash_{n_1} d$, respectively. This describes the splitting of $V_{\underline{\lambda}}$ into irreducible Π -modules with respect to Γ .

For the second step we note that $V_{\mu_i}(\text{GL}(U_i^*)) \simeq V_{\mu_i^*}(\text{GL}(U_i))$, when we view the left hand side as a $\text{GL}(U_i)$ -module via the isomorphism $\text{GL}(U_i) \rightarrow \text{GL}(U_i^*), a_i \mapsto (a_i^*)^{-1}$.

As a consequence of the Littlewood-Richardson rule [31, 11] we obtain (compare [11, Eq. (11), p.149])

$$\dim(V_{\mu_i^*}(\text{GL}(U_i)) \otimes V_{\tilde{\mu}_i}(\text{GL}(U_i)))^{\text{GL}(U_i)} = \begin{cases} 1 & \text{if } \mu_i = \tilde{\mu}_i, \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that

$$\dim(V_{\underline{\lambda}})^H = \sum_{\mu_1, \mu_2, \mu_3} g(\lambda_{12}, \mu_1, \mu_2) g(\lambda_{23}, \mu_2, \mu_3) g(\lambda_{31}, \mu_3, \mu_1)$$

as claimed. \square

6. EXTENSION PROBLEM

In order to advance, we need to study the difference between $S(w)$ and $S^o(w)$. Let W be of format \underline{m} and $w \in W$ be stable. If $\underline{\lambda} \in S^o(w)$, then Proposition 3.8 implies that there exists $k \in \mathbb{Z}$ such that $\underline{\lambda} + k\varepsilon_{\underline{m}} \in S(w)$, where $\varepsilon_{\underline{m}} = (\varepsilon_{m_1}, \varepsilon_{m_2}, \varepsilon_{m_3})$. It is of interest to know the smallest such k . Below we will see that k can be given a geometric interpretation in terms of the problem of extending regular functions from Gw to $\overline{G}w$.

We call the group morphism $\det: G \rightarrow \mathbb{C}^\times, (g_1, g_2, g_3) \mapsto \det g_1 \det g_2 \det g_3$ the determinant on G . In the following we will assume that $\varepsilon_{\underline{m}} \in S^o(w)$. By Proposition 3.4 this is

equivalent to $\det g = 1$ for all $g \in \text{stab}(w)$. We note that this condition is satisfied for $w = \langle n, n, n \rangle$ due to Proposition 5.1.

If $\varepsilon_{\underline{m}} \in S^o(w)$, then \det induces the well defined regular function $\det_w: Gw \rightarrow \mathbb{C}, gw \mapsto \det g$.

LEMMA 6.1. *Let $w \in W \setminus \{0\}$ be stable and $u \in \overline{Gw} \setminus Gw$. Suppose that (g_n) is a sequence in G such that $\lim_{n \rightarrow \infty} g_n w = u$. Then we have $\lim_{n \rightarrow \infty} \det g_n = 0$.*

PROOF. Since $G_s w$ is closed and $0 \notin G_s w$ we have

$$\varepsilon := \inf\{\|\tilde{g}w\| \mid \tilde{g} \in G_s\} = \min\{\|\tilde{g}w\| \mid \tilde{g} \in G_s\} > 0.$$

For each n there are $\tilde{g}_n \in G_s$ such that

$$g_n w = \det g_n \tilde{g}_n w.$$

Hence $\|g_n w\| = |\det g_n| \|\tilde{g}_n w\|$. Since $\lim_{n \rightarrow \infty} \|g_n w\| = \|u\|$ and $\|\tilde{g}_n w\| \geq \varepsilon > 0$ we conclude that $|\det g_n| \leq \|g_n w\|/\varepsilon$ is bounded.

If $\lim_{n \rightarrow \infty} \det g_n = 0$ were false, then there would be some nonzero limit point δ of the sequence $(\det g_n)$. After going over to a subsequence, we have $\lim_{n \rightarrow \infty} \det g_n = \delta$. From (6) we get $\lim_{n \rightarrow \infty} \tilde{g}_n w = \delta^{-1}u$. Hence $\delta^{-1}u \in \overline{G_s u} = G_s u$, which implies the contradiction $u \in Gw$. \square

THEOREM 6.2. *Suppose that $w \in W$ is a stable tensor and $\varepsilon_{\underline{m}} \in S^o(w)$.*

1. *Then w has the cubic format (m, m, m) .*
2. *The extension of \det_w to \overline{Gw} with value 0 on the boundary $\overline{Gw} \setminus Gw$ is continuous in the \mathbb{C} -topology.*
3. *\det_w is not a regular function on \overline{Gw} if $m > 1$.*
4. *\overline{Gw} is not a normal variety if $m > 1$.*
5. *For all highest weight vectors $f \in \mathcal{O}(Gw)$ we have $(\det_w)^k f \in \mathcal{O}(\overline{Gw})$ for some $k \in \mathbb{Z}$.*

PROOF. 1. We have $\det g = 1$ for all $g \in \text{stab}(w)$ since $\varepsilon_{\underline{m}} \in S^o(w)$. But $g = (a \text{id}_{m_1}, b \text{id}_{m_2}, c \text{id}_{m_3})$ is in $\text{stab}(w)$ for any $a, b, c \in \mathbb{C}^\times$ with $abc = 1$. This implies

$$1 = \det g = a^{m_1} b^{m_2} c^{m_3} = a^{m_1 - m_3} b^{m_2 - m_3}.$$

Therefore, $m_1 = m_2 = m_3$.

2. This follows from Lemma 6.1.
3. We note that for $w \in W$ and $g, h \in G$

$$\begin{aligned} g \det_w(hw) &= \det_w(g^{-1}hw) = \det(g^{-1}h) \\ &= \det(g)^{-1} \det(h) = \det(g)^{-1} \det_w(hw). \end{aligned} \quad (6.1)$$

If \det_w had a regular extension to \overline{Gw} , then this shows that $\mathcal{C}\det_w$ is a submodule of $\mathcal{O}(\overline{Gw})$ of highest weight $-\varepsilon_{\underline{m}}$. Hence $\mathcal{O}(W)$ would contain an irreducible submodule of highest weight $-\varepsilon_{\underline{m}}$ as well. On the other hand, the Kronecker coefficient $g(\varepsilon_{\underline{m}})$ vanishes if $m > 1$. This contradicts (3.2). and proves that \det_w is not a regular function on \overline{Gw} .

4. Some standard facts from algebraic geometry [27, III,§8] combined with part 2 and part 3 imply that \overline{Gw} is not a normal variety.

5. This follows by tracing the proof of Proposition 3.8. \square

COROLLARY 6.3. (1) *The orbit closure of the matrix multiplication tensor $\langle n, n, n \rangle$ is not normal if $n > 1$. (2) *The orbit closure of the unit tensor $\langle m \rangle$ is not normal if $m \geq 5$.**

PROOF. (1) The first assertion is immediate from Theorem 6.2.

(2) Put $W = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, $w := \langle m \rangle$. Proposition 4.1 implies that $(\det g)^2 = 1$ for all $g \in \text{stab}(w)$. As in the proof of Theorem 6.2 we show that $\det_w^2: Gw \rightarrow \mathbb{C}$ has a continuous extension to \overline{Gw} .

If \det_w^2 had a regular extension to \overline{Gw} , then $\mathcal{O}(W)$ would contain an irreducible submodule of highest weight $\underline{\lambda} = (2^m, 2^m, 2^m)$ (compare (6.1)). On the other hand, using the symmetry property $g(\lambda, \mu, \nu) = g(\lambda', \mu', \nu)$ of Kronecker coefficients [31] (with λ' denoting the transposed partition), we obtain $g(2^m, 2^m, 2^m) = g(m^2, m^2, 2^m) = 0$ for $m \geq 5$. (The vanishing since the right hand partition has more than four rows.) This contradicts (3.2) and proves that \det_w^2 does not have a regular extension to \overline{Gw} . The assertion follows now with [27, III,§8] as in the proof of Theorem 6.2. \square

The nonnormality of these orbit closures indicates that the extension problem is delicate. Kumar [20] recently obtained similar conclusions for the orbit closures of the determinant and permanent by different methods.

We also make the following general observation.

PROPOSITION 6.4. *Suppose that $w \in W$ is stable. Then $\text{stab}(w)$ is reductive, Gw is affine. Further, $\overline{Gw} \setminus Gw$ is either empty or of pure codimension one in \overline{Gw} .*

7. MOMENT POLYTOPES

Since the semigroups $S(w)$ seem hard to determine, one may take a coarser viewpoint, as already suggested by Strassen [35, Eq. (57)]. We set $\Delta_{\underline{m}} := \Delta_{m_1} \times \Delta_{m_2} \times \Delta_{m_3}$, where $\Delta_m := \{x \in \mathbb{R}^m \mid x_1 \geq \dots \geq x_m \geq 0, \sum_i x_i = 1\}$.

DEFINITION 7.1. *The moment polytope $P(w)$ of a tensor $w \in W$ is defined as the closure of the set $\{\frac{1}{d}\underline{\lambda} \mid d > 0, \underline{\lambda} \in S(w) \cap \Lambda_d^+(\underline{m})\}$.*

Note that $P(w) \subseteq \Delta_{\underline{m}}$ is a polytope since $S(w)$ is a finitely generated semigroup. We have

$$\overline{Gw} \subseteq \overline{Gv} \implies S(w) \subseteq S(v) \implies P(w) \subseteq P(v)$$

Hence exhibiting some point in $P(w) \setminus P(\langle m \rangle)$ would establish the lower bound $\underline{R}(w) > m$.

The moment polytope of a generic tensor w of format \underline{m} equals the *Kronecker polytope* $P(\underline{m})$, which is defined as the closure of $\{\frac{1}{d}\underline{\lambda} \mid \underline{\lambda} \in K(\underline{m}), \underline{\lambda} \in \Lambda_d^+(\underline{m})\}$, compare Lemma 3.2. This complicated polytope has been the object of several recent investigations [1, 18, 29, 4] and $P(\underline{m})$ is by now understood to a certain extent. We remark that the Kronecker polytope $P(\underline{m})$ is closely related to the quantum marginal problem of quantum information theory, cf. [7, 18].

Let $u_m := (1/m, \dots, 1/m) \in \Delta_m$ denote the uniform distribution and consider the vertex $u_{\underline{m}} := (u_m, u_m, u_m)$ of the polytope $\Delta_{(m,m,m)}$. The following follows, e.g., from [35, Satz 11].

LEMMA 7.2. *We have $u_{\underline{m}} \in P(w)$ both for $w = \langle m \rangle$ and $w = \langle n, n, n \rangle$, $m = n^2$.*

Resolving the following question seems of great relevance.

PROBLEM 7.3. *Determine the moment polytopes of unit tensors and matrix multiplication tensors.*

Replacing $S(w)$ by $S^\circ(w)$ in the definition of $P(w)$ we obtain the larger polytope $P^\circ(w)$.

THEOREM 7.4. *We have $P^\circ(\langle m \rangle) = \Delta_{(m,m,m)}$ and $P^\circ(\langle n, n, n \rangle) = \Delta_{(n^2, n^2, n^2)}$.*

PROOF. The statement for the unit tensors is an easy consequence of Corollary 4.5(2).

For the second statement take any $\underline{\lambda} = (\lambda_{12}, \lambda_{23}, \lambda_{31})$ in $\Lambda_{dn}^+(n^2, n^2, n^2)$. Consider the rectangular partition $(d^n) = (d, \dots, d) \vdash_n dn$. The main result in [4] states that for $ij = 12, 23, 31$ there exists a positive stretching factor $k_{ij} \in \mathbb{N}$ such that $g(k_{ij}\lambda_{ij}, (k_{ij}d)^n, (k_{ij}d)^n) \neq 0$. Let k be the least common multiple of k_{12}, k_{23}, k_{31} . Then we have for $ij = 12, 23, 31$

$$g(k\lambda_{ij}, (kd)^n, (kd)^n) \neq 0.$$

Theorem 5.3 with $\mu_i = (kd)^n \vdash_n kdn$ implies that $k\underline{\lambda} \in S^\circ(\langle n, n, n \rangle)$. Hence $\frac{1}{dn}\underline{\lambda} \in P^\circ(\langle n, n, n \rangle)$. Since the set of $\frac{1}{dn}\underline{\lambda}$ is dense in $\Delta(n^2, n^2, n^2)$, we obtain $P^\circ(\langle n, n, n \rangle) = \Delta_{(n^2, n^2, n^2)}$ as claimed. \square

The following is a consequence of Proposition 3.8.

LEMMA 7.5. *Let w be stable and suppose that $u_{\underline{m}} \in P(w)$. Then there exists $\delta > 0$ such that for all $\underline{x} \in P^\circ(w)$ and all $0 \leq t \leq \delta$ we have $t\underline{x} + (1-t)u_{\underline{m}} \in P(w)$.*

By combining Theorem 7.4 with Lemma 7.2, Lemma 7.5, and the stability of the unit and matrix multiplication tensors, we obtain the following result.

COROLLARY 7.6. *For both $w = \langle m \rangle$ and $w = \langle n, n, n \rangle$, $m = n^2$, there is an open neighborhood U of $u_{\underline{m}}$ such that $U \cap P(w) = U \cap \Delta_{\underline{m}}$. In particular, $\dim P(w) = \dim \Delta_{\underline{m}}$.*

8. EXAMPLES AND COMPUTATIONS

8.1 A family of G -obstructions

We use the frequency notation $k_1^{e_1} k_2^{e_2} \dots k_s^{e_s}$ to denote the partition of $\sum_i k_i e_i$ where k_i occurs e_i times. Consider the highest weights $\underline{\lambda}_n := (2^{n^2} 0, 2^{n^2} 0, (2n^2 - 3)^1 1^3 0^{n^2-3})$ for $n \geq 2$.

LEMMA 8.1. *1. $\underline{\lambda}_n \notin S^\circ(\langle n^2 + 1 \rangle)$.*

2. $\underline{\lambda}_n \in S(\langle n, n, n \rangle)$.

3. $\underline{R}(\langle n, n, n \rangle) > n^2 + 1$.

PROOF. (Outline) 1. Put $m := n^2$ and $\underline{\lambda}_n = (\lambda_1, \lambda_2, \lambda_3)$. According to Proposition 3.4 and Theorem 4.4 we need to show that $(V_{\lambda_1}^\alpha \otimes V_{\lambda_2}^\alpha \otimes V_{\lambda_3}^\alpha)^{\text{stab}(\alpha)} = 0$ for all $\alpha \in \text{Par}_{m+1}(2m)$ smaller than $\lambda_1, \lambda_2, \lambda_3$. There are only two such partitions α , namely, $2^m 0$ and $2^{m-1} 1^2$, and in (8.2) and (8.3) below we shall prove that indeed $(V_{\lambda_1}^\alpha \otimes V_{\lambda_2}^\alpha \otimes V_{\lambda_3}^\alpha)^{\text{stab}(\alpha)} = 0$ for those α .

2. Using [28, 30] one can show $g(\underline{\lambda}_n) = 1$. Hence the highest weight vector $f \in \mathcal{O}(W)$ of weight $\underline{\lambda}_n$ is uniquely determined up to a scalar. We explicitly constructed f and (guided by computer calculations) proved that $f(\langle n, n, n \rangle) \neq 0$. Hence $\underline{\lambda}_n \in S(\langle n, n, n \rangle)$.

3. This follows from the first two parts and (3.1). \square

REMARK 8.2. *1. Theorem 5.3 with $\mu_i = (2n)^n$, a positivity proof for the resulting Kronecker coefficients (Lemma 8.6),*

and Proposition 3.4 yield $\underline{\lambda}_n \in S^\circ(\langle n, n, n \rangle)$. In order to guarantee $\underline{\lambda}_n \in S(\langle n, n, n \rangle)$ we currently know of no better way than to evaluate a highest weight vector at some point in $G\langle n, n, n \rangle$. In general, this becomes prohibitively costly for larger dimension formats.

2. Lemma 8.1 yields $\underline{R}(\langle 2, 2, 2 \rangle) > 5$. It is known [21] that $\underline{R}(\langle 2, 2, 2 \rangle) = 7$. So far we have been unable to reach the optimal lower bound by an obstruction.

8.2 Strassen's invariant

Let $W = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^3$, $m \geq 3$, and consider $\underline{\lambda}_m := (3^m, 3^m, m^3)$. Strassen [33] constructed an explicit invariant $f_m \in \mathcal{O}(W)$ of highest weight $\underline{\lambda}_m^*$, that vanishes on all tensors in W with border rank at most $r = \lceil 3m/2 \rceil - 1$. Hence $f_m(w) \neq 0$ implies $\underline{R}(w) > r$.

Let $\underline{\lambda}'_m \in \Lambda^+(r, r, r)$ be obtained from $\underline{\lambda}_m$ by appending zeros. It is tempting to conjecture that $\underline{\lambda}'_m \notin S(\langle r \rangle)$, because then Strassen's implication would be a consequence of the existence of the obstruction $\underline{\lambda}_m$. Indeed, $f_m(w) \neq 0$ implies $\underline{\lambda}_m \in S(w)$ and, assuming the conjecture, $\underline{\lambda}_m \in S(w) \setminus S(\langle r \rangle)$ and thus $\underline{R}(w) > r$. Unfortunately, the conjecture is already false for $m = 4$! An extensive computer calculation revealed the existence of $\tilde{f}_4 \in \mathcal{O}(W)_{12}$ of highest weight $\underline{\lambda}'_4$ and $g \in G$ such that $\tilde{f}_4(g(5)) \neq 0$, which shows $\underline{\lambda}'_4 \in S(\langle 5 \rangle)$. Note $g(\underline{\lambda}_4) = 2$.

8.3 Explicit Schur-Weyl modules

In the remainder of the paper we explain the mathematics that allows the explicit computations of weight spaces V_λ^α .

For the following well known facts see [11, 12]. Let $V = \mathbb{C}^m$ with the standard basis e_1, \dots, e_m . For a partition $\lambda \vdash_m d$ we denote by $\mathcal{T}_m(\lambda)$ the set of tableaux T of shape λ with entries in $\{1, 2, \dots, m\}$. Every $T \in \mathcal{T}_m(\lambda)$ has a content $\alpha \in \mathbb{N}^m$, where α_j counts the number of occurrences of j in T .

Let St_λ denote the standard tableau arising when we number the boxes of the Young diagram of λ columnwise downwards, starting with the leftmost column. We assign to $T \in \mathcal{T}_m(\lambda)$ the basis vector $e(T) := e_{j_1} \otimes \dots \otimes e_{j_d} \in V^{\otimes d}$, where $j_k \in \{1, \dots, m\}$ is the entry of T at the box which is numbered k in St_λ . In other words, j_k is the k th entry of T when we read the tableau T columnwise downwards, starting with the leftmost column. Note that $e(T)$ is a weight vector with respect to the subgroup $T_m \subseteq \text{GL}_m$ of diagonal matrices, and the weight of $e(T)$ equals the content α of T . One should think of the tableau T as a convenient way to record the basis vector $e(T)$.

Let P_λ the subgroup of permutations in St_d preserving the rows of St_λ and denote by Q_λ the subgroup of permutations in St_d preserving the columns of S_λ . To any $T \in \mathcal{T}_m(\lambda)$ we assign now the following vector in $V^{\otimes d}$:

$$v(T) := \frac{1}{|P_\lambda||Q_\lambda|} \sum_{\substack{\sigma \in Q_\lambda \\ \pi \in P_\lambda}} \text{sgn}(\sigma) \sigma \pi e(T).$$

Note that $v(T)$ is a weight vector and its weight equals the content α of T .

Let T_λ denote the semistandard tableau which in the i th row only has the entry i . Clearly, T_λ has the content λ . Let $\mu = \lambda'$ denote the partition dual to λ and let ℓ be the length of μ . Then we have

$$v_\lambda := v(T_\lambda) = (e_1 \wedge \dots \wedge e_{\mu_1}) \otimes \dots \otimes (e_1 \wedge \dots \wedge e_{\mu_\ell}). \quad (8.1)$$

It is easy to see that v_λ is a U_m -invariant weight vector of weight λ , where $U_m \subseteq \text{GL}_m$ denotes the subgroup of upper triangular matrices with ones on the main diagonal. Hence the GL_m -submodule V_λ generated by v_λ is irreducible of highest weight λ . We have

$$V_\lambda = \text{span}\{(x_1 \wedge \dots \wedge x_{\mu_1}) \otimes \dots \otimes (x_1 \wedge \dots \wedge x_{\mu_\ell}) \mid x_i \in V\}$$

and V_λ is spanned by $\{v(T) \mid T \in \mathcal{T}_m(\lambda)\}$. It is well known that the $v(T)$ form a basis of V_λ when T runs over all semistandard tableaux in $\mathcal{T}_m(\lambda)$. A basis of the weight space V_λ^α is provided by the $v(T)$ where $T \in \mathcal{T}_m(\lambda)$ runs over all semistandard tableaux with content α . We embed S_m into GL_m by mapping $\pi \in S_m$ to the permutation matrix P_π . Note that the group S_m acts on $\mathcal{T}_m(\lambda)$ by permutation of the entries of the tableaux. Then we have $P_\pi v(T) = v(\pi T)$ for $T \in \mathcal{T}_m(\lambda)$. We have thus found explicit realizations of Schur-Weyl modules.

Suppose now $d = m$ and consider the weight space $\mathcal{S}_\lambda := V_\lambda^{\varepsilon_m}$ of weight $\varepsilon_m = (1, \dots, 1)$. Then \mathcal{S}_λ is a S_m -submodule of V_λ . Moreover, the vectors $v(T)$, where T runs over all standard tableaux of shape λ , provide a basis of \mathcal{S}_λ . One can show that \mathcal{S}_λ is irreducible and isomorphic to $[\lambda]$. We have thus also found an explicit realization of the irreducible S_m -module $[\lambda]$.

8.4 Tableaux straightening

An explicit description of the action the subgroup S_m of GL_m on the basis $(v(T))$ is provided by the following tableau straightening algorithm [11, p.97-99, p.110]. It takes as input any tableau $T \in \mathcal{T}_m(\lambda)$ and expresses the vector $v(T)$ as an integer linear combination of the basis vectors $v(S)$, where S is semistandard. This way, we obtain an explicit description of the operation of $\text{stab}(\alpha)$ on the weight space V_λ^α , which is required for applying Theorem 4.4.

1. If T is semistandard, return $v(T)$.
2. If the columns of T do not have pairwise distinct entries, return 0. Otherwise, apply column permutations π to put all columns in strictly increasing order by applying the rule $\pi v(T) = \text{sgn}(\pi) v(\pi T)$.
3. If the resulting tableau is not semistandard, suppose the k th entry of the j th column is strictly larger than the k th entry of the $(j+1)$ th column. Then we have $v(T) = \sum_S v(S)$, where S ranges over all tableaux that arise from T by exchanging the top k elements from the $(j+1)$ th column with any selection of k elements in the j th column, preserving their vertical order. Continue recursively with the resulting S .

See [11, p. 110] for a proof that this algorithm terminates (whatever choice of k and j is made in step (3)).

8.5 Details of the proof of Lemma 8.1

Recall from §4.2 that the weight space V_λ^α is invariant under the action of $\text{stab}(\alpha)$. We are interested in the splitting of V_λ^α into irreducible $\text{stab}(\alpha)$ -modules.

REMARK 8.3. *In the special case $\alpha = d\varepsilon_m$, where $\text{stab}(\alpha) = S_m$, it is known [13] that the arising multiplicities are special plethysm coefficients, namely*

$$\text{mult}([\pi], V_\lambda^{d\varepsilon_m}) = \text{mult}(V_\lambda(\text{GL}_m), S_\pi(\text{Sym}^d \mathbb{C}^m)) \text{ for } \pi \vdash m.$$

LEMMA 8.4. *Let $0 \leq s < m$, $d \geq 1$. Then $V_{(dm-s)1^s}^{d\varepsilon_m} \simeq [(m-s)1^s]$ as S_m -modules.*

PROOF. Let \mathcal{ST} denote the set of semistandard tableaux of shape $(dm-s)1^s$ and content $d\varepsilon_m$. Moreover, let \mathcal{S} denote the set of standard tableaux of shape $(m-s)1^s$. Let $T \in \mathcal{ST}$ and suppose that $1, a_1, \dots, a_s$ are the entries of the first column of T . After deleting $d-1$ of the boxes with the entries $1, \dots, m$ from the first row of T , we obtain a standard tableau $\psi(T) \in \mathcal{S}$. It is clear that $\psi: \mathcal{ST} \rightarrow \mathcal{S}$ is a bijection. The algorithmic description of the Schur-Weyl modules above easily implies that $\psi(\pi T) = \pi\psi(T)$ for any $\pi \in S_m$. We use now that $(v(T))_{T \in \mathcal{ST}}$ and $(v(T'))_{T' \in \mathcal{S}}$ form a basis of the weight space $V_{(dm-s)1^s}^{d\varepsilon_m}$ and the S_m -module $[(m-s)1^s]$ realized as a submodule of $(\mathbb{C}^m)^{\otimes m}$ as in §8.3. \square

Taking $d = 2$ and $s = 3$ we get from Lemma 8.4

$$(V_{2\varepsilon_m}^{2\varepsilon_m} \otimes V_{2\varepsilon_m}^{2\varepsilon_m} \otimes V_{(2m-3)1^3}^{2\varepsilon_m}) \simeq ([m] \otimes [m] \otimes [(m-3)1^3])^{S_m} = 0 \quad (8.2)$$

which was needed in the proof of the first part of Lemma 8.1.

LEMMA 8.5. *As $S_{m-1} \times S_2$ -modules we have*

1. $V_{(2^m)}^{2^{m-1}1^2} \simeq [m-1] \otimes [2]$.
2. $V_{(2m-3)1^3}^{2^{m-1}1^2} \simeq ([(m-4)1^3] \otimes [2]) \oplus ([(m-2)1] \otimes [1^2]) \oplus ([(m-3)1^2] \otimes [2]) \oplus ([(m-3)1^2] \otimes [1^2])$.

PROOF. 1. There is a single semistandard tableau T of shape 2^m and content $2^{m-1}1^2$: the i th row contains the entries i, i for $i < m$ and the m th row contains $m, m+1$. The tableau T is fixed by the action of S_{m-1} . The transposition π in S_2 exchanges m and $m+1$. However, the straightening algorithm shows that $v(\pi T) = v(T)$.

2. The basis of $V_{(2m-3)1^3}^{2^{m-1}1^2}$ is indexed by semistandard tableaux which fall into four different classes as indicated below, where $p := m-1$ and $q := m+1$:

$\begin{array}{cccccccc} 1 & 1 & 2 & 2 & 3 & 3 & \dots & p & p & q \\ a & & & & & & & & & \\ b & & & & & & & & & \\ c & & & & & & & & & \end{array}$	Class 1	$\begin{array}{cccccccc} 1 & 1 & 2 & 2 & 3 & 3 & \dots & p & p \\ a & & & & & & & & \\ m & & & & & & & & \\ q & & & & & & & & \end{array}$	Class 2
$\begin{array}{cccccccc} 1 & 1 & 2 & 2 & 3 & 3 & \dots & p & p & q \\ a & & & & & & & & & \\ b & & & & & & & & & \\ m & & & & & & & & & \end{array}$	Class 3	$\begin{array}{cccccccc} 1 & 1 & 2 & 2 & 3 & 3 & \dots & p & p & m \\ a & & & & & & & & & \\ b & & & & & & & & & \\ q & & & & & & & & & \end{array}$	Class 4

Omitting the boxes with entries $m, m+1$ in the tableaux of class 1 and deleting repeated entries in the first row, we obtain a bijection of the set of tableaux of class 1 with the set of standard tableaux of shape $(m-4)1^3$. It follows that the span of the basis vectors of class 1 is isomorphic to $[(m-4)1^3] \otimes [2]$.

Similarly, the tableaux of class 2 are in bijection with the standard tableaux of the shape $(m-2)1^3$. The span of the basis vectors of class 2 is isomorphic to $[(m-2)1^3] \otimes [1^2]$ (note the sign change when permuting m with $m+1$).

Let \mathcal{T} denote the set of tableaux of class 3 and consider the transposition $\pi := (m m+1)$. Then $\{\pi T \mid T \in \mathcal{T}\}$ is the set of tableaux of class 4. Clearly, \mathcal{T} is in bijection with the standard tableaux of shape $(m-3)1^2$. The vectors $v(T) + \pi v(T)$ for $T \in \mathcal{T}$ span $[(m-3)1^2] \otimes [2]$, whereas the vectors $v(T) - \pi v(T)$ span $[(m-3)1^2] \otimes [1^2]$. \square

Lemma 8.5 implies now

$$(V_{(2^m)}^{2^{m-1}1^2} \otimes V_{(2^m)}^{2^{m-1}1^2} \otimes V_{(2m-3)1^3}^{2^{m-1}1^2})^{S_{m-1} \times S_2} = 0, \quad (8.3)$$

which was needed in the proof of the first part of Lemma 8.1. The following was claimed in Remark 8.2.

LEMMA 8.6. Let $n \geq 2$. Then $g(2^{n^2}, (2n)^n, (2n)^n) = 1$ and $g((2n^2 - 3)1^3, (2n)^n, (2n)^n) > 0$.

PROOF. The first claim follows from [8, Satz 3.1]. For the second claim, we note that $\underline{\lambda} := ((2n^2 - 3)1^3, (2n)^n, (2n)^n)$ can be decomposed as $\underline{\lambda} = \underline{\mu} + (n - 1) \cdot (2n, 2^n, 2^n)$, where $\underline{\mu} := ((2n - 3)1^3, 2^n, 2^n)$. It is clear that $g(2n, 2^n, 2^n) = 1$. It follows from [28, 30] that $g(\underline{\mu}) = g((2n - 3)1^3, n^2, n^2) = 1$. Since the triples with positive Kronecker coefficients form a semigroup, the second assertion follows. \square

9. REFERENCES

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